Collateral requirements, capital flows, and welfare: A general equilibrium approach*

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June 29, 2017

Abstract

We build a general equilibrium with agents being heterogeneous in term of savings and productivities and financial market imperfections in the form of collateral constraints. In this framework, we firstly examine how collateral requirements affect the magnitude and direction of capital flows among agents. Second, we investigate the correlation between the collateral constraints and welfare of the lender and borrower as well as the aggregate welfare. We then characterize the optimal collateral constraints which maximize the social welfare functions measured by different ways.

JEL Classifications: D53, E44, G10.
Keywords: General equilibrium, collateral constraints, financial frictions, capital flows, welfare.

*The authors would like to thank Cuong Le-Van and Ha-Huy Thai for their helpful comments and suggestions.

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1 Introduction

In an environment where there exist frictions of contract enforcement, collateralized debts arise naturally for the lender to secure his loans. Such collateral requirements are considered as endogenous borrowing constraints, which depends on the values of the assets and also the possible losses associated with the reallocation of those assets in case of default. This source of financial friction has been of great interest to both theoretical and empirical macroeconomists, mostly in examining the role of collateral constraints in the dynamic of the business cycles since the seminal paper of Kiyotaki and Moore (1997). The literature of the impact of collateral constraints on welfare of the borrower and the lender as well as the social welfare is, however, relatively new. To the best of my knowledge, there is no theoretical research focusing on the correlation between collateral constraints and welfare so far. Inspired by this literature gap, we construct a two-agent, two-period general equilibrium model with heterogeneous agents (in terms of savings and productivities) and financial frictions (in the form of collateral requirements) to examine whether collateral constraints always reduce the social welfare and agents' consumption. Following the lead of Matsuyama (2005), Quadrini (2011), and Bosi et al. (2017), we assume that, due to imperfections in the financial market, agents can borrow an amount whose repayment is at most a fraction of their future output. In this framework, we aim to figure out how the strictness of the collateral constraints influence the agents' and the social welfare.

Closely related to our research interest is the paper of Obiols-Homs (2011), which studies the effects on welfare of exogenous borrowing limits. As mentioned in his paper, the usual economic intuition would advocate loosening tight borrowing constraints, since they may prevent achieving fully efficient allocations among agents of different productivities. Opposite to this view, Obbiols-Homs shows that too large borrowing limits may have negative effects on welfare. In particular, in a two-period model of exchange with an infinite number of agents and fixed borrowing limits, he concludes that the lender always gains with an increase in the borrowing limit, whereas, borrowers may lose if the initial borrowing limit is sufficiently large. Furthermore, the aggregate welfare in equilibrium (or ex ante welfare) displays an inverted U-shape as a function of the borrowing limit.

In an economy with endogenous borrowing constraints, we reach similar conclusions on the effects of loosening the borrowing constraints on the wealth of the lender and borrower. As regards the social welfare, our contribution is to provide a more general picture, demonstrating that the impact of raising the borrowing limit on the aggregate welfare differs through different ways of aggregating welfare and choosing individual utility functions. For instance, we show that the aggregate welfare, measured as an unweighted sum of consumptions of agents, is an increase function of the borrowing limit as long as the borrowing constraints are still binding. However, if we measure the aggregate welfare as a weighted sum over agents' consumptions, then whether the aggregate welfare is presented by an inverted U-shape or an increasing function of the borrowing limit depends on which agent's consumption is assigned a higher weight. In particular, the former occur if the weight for the borrower's consumption is higher than that for the lender's consumption, and the latter happens otherwise. Furthermore, we also give clear comparisons on general equilibrium outcomes including interest rate, agents' wealth, and the aggregate output in a collaterally constrained economy and those in a collaterally unconstrained . We conclude that the the interest rate is higher in an unconstrained economy, and the same for the lender's wealth and the aggregate output. For the borrower, his wealth is lower in an unconstrained economy, at least when the collateral constraint imposed on the borrower is loosen over a certain threshold. In a simple framework where the production is assumed linear, we manage to solve explicitly the optimal collateral constraint and show that the higher weight assigned to the lender's utility in the social welfare, the looser the optimal collateral should be.

The second purpose of this paper is to investigate the capital flows in an economy with collateral constraints. For this purpose, the borrower (lender) in our model can be interpreted as a country having capital inflows (outflows). In the absence of financial frictions, the neoclassical growth models suggest that countries which are more productive should invest more and attract more foreign capital. Our findings from the benchmark model where there are no collateral constraints confirm the prediction of classical models. Under the presence of asymmetry in financial development, the conclusions drawn from the theoretical literature are not very consistent across papers. On one hand, as a review by Ikeda and Phan (2014), a large friction of the literature shows that the global asymmetry in financial frictions causes uphill capital flows, which is contrary to the finding of neoclassical growth models, such as, Chari et al. (2007) Caballero et al. (2007), Song et al. (2011). On the other hand, Matsuyama (2005), for examples, shows that in a two-country economy where countries differ in endowments and collateral rules, the direction of capital flows could coincide with neoclassical growth models' prediction if the asymmetry in collateral rules is not so huge. Also examining the impact of collateral constraints, we find that collateral constrains affect the magnitude, while has no impact on the direction of the capital flows in a framework with two countries being heterogeneous in terms of productivity, endowment, and credit friction. When extending this framework to include three countries having three levels of productivities, we find that while the most productive country always borrows, the least productive country always lends, the one with an intermediate level of productivity borrows if and only if its productivity is over a certain threshold. This threshold is determined by the difference in savings between the most and the least productive agent, productivity and collateral constraint imposed on the least productive. In particular, the country of intermediate level of productivity has capital inflows when the saving of the least productive country is very high and/or the most productive country suffers from strict collateral constraint. This country, on the other hand, has capital outflows if saving of the least productive country is not sufficiently high and/or the productivity is very high, and the collateral constraint is loose in the most productive country.

The rest of this paper is organized as follows. In section 2, we introduce the benchmark model without collateral constraints and provide some preliminary equilibrium properties. In section 3, we incorporate the collateral constraints into the benchmark model and study the impact of such constraints on equilibrium outcomes including the interest rate, outputs, and welfare. In section 4, we examine the capital flows in a collaterally constrained world with three countries. Section 5 concludes. All technical proofs are gathered in Appendices.

2 The benchmark model: A two-agent economy without financial frictions

We consider a two-period economy with two agents. There is no uncertainty and there is a single good which can be consumed or used to produce. Each agent i has exogenous saving (S_i units of good) at the initial date and decides how much good for production and investment in the financial market in order to maximize his wealth in the next period. Since each agent lives for two periods, this wealth is also the agent's consumption.

On one hand, if agent i buys k_i units of physical capital, she will produce $F(k_i)$ units of good at the second date, where F_i is her production function.

On the other hand, she can invest in financial assets with real return r. Denote a_i the amount that the agent i invests in financial asset. She can also borrow and then pay back ra_i in the next period. Assume that there is no borrowing constraint.

The economy, denoted by $\mathcal{E}$, is characterized by a list of fundamentals:

$$\mathcal{E} \equiv (F_i, S_i)_{i=1,2}$$

Definition 1. *Given the economy $\mathcal{E}$, a list (r, a_1, a_2, k_1, k_2) is an equilibrium of the two-agent, non borrowing constraint economy if the following conditions are satisfied:*

(i) Agents' optimality: for each agent i ($i \in \{1, 2\}$), given r, (a_i, k_i) solves his consumption maximization problem.

$$(P_i) \qquad c_i = \max_{k_i, a_i} [F_i(k_i) - ra_i] \tag{1}$$

$$\textit{subject to}: \qquad 0 \leq k_i \leq S_i + a_i \tag{2}$$

($a_i < 0$ mean that the agent i lends; $a_i > 0$ means that the agent i borrows).

(ii) Financial market clearing condition:

$$\sum_{i=1,2} a_i = 0$$

2.1 Linear technology

Assume that $F_i(k_i) = A_i k_i$ for any i, the problem (P_i) becomes:

$$(P_i^L) \qquad c_i = \max_{k_i, a_i} [A_i k_i - ra_i]$$

$$\text{subject to}: \qquad 0 \leq k_i \leq S_i + a_i$$

Lemma 1 (Individual problem)**.** *The solution (a_i, k_i) of the agent i's maximization problem with linear technology and no borrowing constraint are determined as follows:*

1. *If $A_i > r$: $a_i = +\infty$, $k_i = +\infty$*
2. *If $A_i < r$: $a_i = -S_i$, $k_i = 0$*
3. *If $A_i = r$: $c_i = AS_i$ for any pairs (a_i, k_i) that satisfies $a_i \geq -S_i$ and $k_i = a_i + S_i$.*

Proof. See Appendix 6.1. □

Thus, when the productivity is low, in the sense that $A_i < r$, the agent i invests all his saving in the financial market and produces nothing. When the productivity is high, i.e $A_i > r$, the agent i borrows as much as he can (which is an infinite amount in this case where no borrowing limits are imposed).

From Lemma 1, we see that at the equilibrium, $r \geq max\{A_1, A_2\}$.

Assumption 1. $A_1 > A_2$.[1]

In this simple economy, it is easy to compute equilibrium.

[1] We are not interested by the case $A_1 = A_2$ because it is not generic.

Proposition 1 (Equilibrium)**.** *Under Assumption 1, there exists a unique equilibrium determined by:*

$$\begin{aligned} \textit{Interest rate:} \quad & r = A_1 \\ \textit{Capital allocation:} \quad & k_1 = S_1 + S_2, \quad k_2 = 0 \\ \textit{Financial assets:} \quad & a_1 = S_2, \quad a_2 = -S_2 \end{aligned}$$

The aggregate output and consumption of each agent are:

$$\begin{aligned} &Y = A_1(S_1 + S_2), \\ &c_1 = A_1 S_1, \quad c_2 = A_1 S_2. \end{aligned}$$

Thus, we see that at equilibrium, the agent with lower productivity will lend all his saving to the more productive agent. The equilibrium interest rate equals to the marginal productivity of the more productive agent.

2.2 Cobb-Douglas technology

We now assume that $F(k_i) = A_i k_i^\alpha$ for each $i = 1, 2$. The problem (P_i) becomes:

$$\begin{aligned} (P_i^C) \qquad & c_i = \max_{k_i, a_i} [A_i k_i^\alpha - r a_i] \\ \text{subject to :} \qquad & 0 \leq k_i \leq S_i + a_i \end{aligned}$$

Since $F'(0) = +\infty$, we have $k_i > 0$ at opimum.

Lemma 2 (Individual problem)**.** *The solution (a_i, k_i) of the agent i's maximization problem with Cobb-Douglas technology and no borrowing constraint is determined by:*

$$k_i = \left(\frac{\alpha A_i}{r}\right)^{\frac{1}{1-\alpha}} \quad \textit{and} \quad a_i = k_i - S_i.$$

Proof. See Appendix 6.2 □

Proposition 2 (Equilibrium)**.** *With Cobb-Douglass production function and no borrowing constraints, there exists a unique equilibrium, which is given by:*

$$\begin{aligned} \textit{Interest rate:} \quad & r = \bar{r} \equiv \frac{\alpha\big(A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}\big)^{1-\alpha}}{(S_1 + S_2)^{1-\alpha}} \\ \textit{Allocation:} \quad & k_i = \left(\frac{\alpha A_i}{r}\right)^{\frac{1}{1-\alpha}} = \frac{A_i^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}(S_1 + S_2), \quad a_i = k_i - S_i \end{aligned}$$

At equilibrium, agent i borrows if and only if his lowest marginal productivity in autarky $(\alpha A_i S_i^{\alpha-1})$ is greater than the other's.

The aggregate output and consumption of each agent are:

$$\begin{aligned} Y &= \bar{Y} \equiv (S_1 + S_2)^\alpha \Big((A_1)^{\frac{1}{1-\alpha}} + (A_2)^{\frac{1}{1-\alpha}}\Big)^{1-\alpha} \\ c_1 &= \bar{c}_1 \equiv \left(\frac{S_1 + S_2}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}\right)^\alpha A_1^{\frac{1}{1-\alpha}} - \frac{\alpha(A_1^{\frac{1}{1-\alpha}} S_2 - A_2^{\frac{1}{1-\alpha}} S_1)}{(S_1 + S_2)^{1-\alpha}\big(A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}\big)^\alpha} \\ c_2 &= \bar{c}_2 \equiv \left(\frac{S_1 + S_2}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}\right)^\alpha A_2^{\frac{1}{1-\alpha}} - \frac{\alpha(A_2^{\frac{1}{1-\alpha}} S_1 - A_1^{\frac{1}{1-\alpha}} S_2)}{(S_1 + S_2)^{1-\alpha}\big(A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}\big)^\alpha} \end{aligned}$$

3 A two-agent economy with collateral constraints

The framework we consider now is similar to that in the case under no borrowing constraints, except that the financial market now is imperfect under which each agent i cannot borrow more than an exogenous fraction $f_i < 1$ of his project's income $F(k_i)$.

Thus, the maximization problem of agent i becomes:

$$(P'_i) \qquad c_i = \max_{k_i, a_i} [F_i(k_i) - r a_i] \tag{3}$$

$$\text{subject to :} \qquad 0 \leq k_i \leq S_i + a_i \tag{4}$$

$$r a_i \leq f_i F_i(k_i) \tag{5}$$

The economy, denoted by $\mathcal{E}'$, is characterized by a list of fundamentals:

$$\mathcal{E}' \equiv (F_i, f_i, S_i)_{i=1,2}$$

Borrowing constraint (5) means that agent i can borrow an amount whose repayment does not exceed an exogenous share of his income. It can be interpreted as the collateral requirement of agent i. In case of default, lenders can seize the fraction f_i of agent i's income. The parameter f_i is set by law and below 1, capturing the possible looses associated with the relocation of wealth in case of default (see Quadrini (2011) for a review of this issue).

Definition 2. *Given the economy $\mathcal{E}'$, a list (r, a_1, a_2, k_1, k_2) is an equilibrium if the following conditions are satisfied:*

1. *Agents' optimality: for each $i \in \{1, 2\}$, given r, (a_i, k_i) is a solution of the problem (P'_i).*
2. *Financial market clearing: $a_1 + a_2 = 0$.*

Assumption 2. $f_i \in [0, 1)$

3.1 Linear technology

With $F_i(k_i) = A_i k_i$ for any i, the problem (P'_i) becomes:

$$(P'^L_i) \qquad c_i = \max_{k_i, a_i} [A_i k_i - r a_i] \tag{6}$$

$$\text{subject to :} \qquad 0 \leq k_i \leq S_i + a_i \tag{7}$$

$$r a_i \leq f_i A_i k_i \tag{8}$$

Suppose $r \leq f_i A_i < A_i$, then the constraint (8) is always satisfied, (7) implies that $k_i = S_i + a_i$, the objective function becomes: $(A_i - r)a_i + A_i S_i$. Consequently, the agent i will choose $a_i = +\infty$ and $k_i = +\infty$

Therefore, $r > f_i A_i$

Lemma 3 (Individual problem)**.** *Under Assumption 1, the solutions for agent i's maximization problem are described as follows.*

1. *If $A_i < r$, then agent i does not produce goods and invest all his saving in the international financial market.*

$$k_i = 0, \quad a_i = -S_i.$$

2. *If $A_i > r > f_i A_i$, then agent i borrows from the financial market and the borrowing constraint is binding.*

$$k_i = \frac{r}{r - f_i A_i} S_i, \quad a_i = \frac{f_i A_i}{r - f_i A_i} S_i$$

3. *If $A_i = r$, then the solutions for the agent's problem include all sets (k_i, a_i) such that $-S_i \leq a_i \leq f_i k_i$ and $k_i = a_i + S_i$.*

Proof. See Appendix 6.3 □

Therefore, we draw the same conclusion about the relationship between productivity and identification of borrower/lender as in the case without borrowing constraint. An agent borrows from the financial market if and only if his productivity is high enough, in the sense that $A_i > r$. Moreover, he borrows the maximum level imposed on him, i.e, the borrowing constraint is binding.

The impact of the collateral requirements on the agent's decision and his consumption is described in the following result.

Corollary 1. *Assume that $A_i > r > f_i A_i$, at optimum:*

$$\frac{\partial k_i}{\partial f_i} > 0, \quad \frac{\partial^2 k_i}{\partial^2 f_i} < 0$$
$$\frac{\partial a_i}{\partial a_i} > 0, \quad \frac{\partial^2 a_i}{\partial^2 f_i} < 0$$
$$\frac{\partial c_i}{\partial f_i} > 0, \quad \frac{\partial^2 c_i}{\partial^2 f_i} < 0$$

Proposition 3 (Equilibrium)**.** *Under Assumptions 1 and 2, there are three cases, each having a unique equilibrium and with agent 1 being the borrower in any cases.*

1. *If $f_1 \leq \frac{A_2}{A_1} \frac{S_2}{S_1 + S_2}$, then the borrowing constraint of agent 1 is binding and there exists an equilibrium characterized by:*

$$\begin{aligned}
&\textit{Interest rate:} \quad r = A_2 \\
&\textit{Physical capital:} \quad k_1 = \frac{A_2}{A_2 - f_1 A_1} S_1, \quad k_2 = -\frac{f_1 A_1}{A_2 - f_1 A_1} S_1 + S_2 \\
&\textit{Financial asset:} \quad a_1 = \frac{f_1 A_1}{A_2 - f_1 A_1} S_1, \quad a_2 = -\frac{f_1 A_1}{A_2 - f_1 A_1} S_1;
\end{aligned}$$

The aggregate output and consumption of each agent are:

$$Y = A_2 S_2 + A_1 S_1 \frac{A_2 - f_1 A_2}{A_2 - f_1 A_1},$$
$$c_1 = A_1 S_1 \frac{A_2 - f_1 A_2}{A_2 - f_1 A_1}, \quad c_2 = A_2 S_2.$$

2. *If* $\frac{A_2}{A_1}\frac{S_2}{(S_1+S_2)} < f_1 < \frac{S_2}{S_1+S_2}$*, then the borrowing constraint of agent 1 is binding, and there exists an equilibrium characterized by:*

$$\textit{Interest rate:} \quad r = f_1 A_1\Big(1 + \frac{S_1}{S_2}\Big) \tag{9}$$

$$\textit{Physical capital:} \quad k_1 = S_1 + S_2, \quad k_2 = 0 \tag{10}$$

$$\textit{Financial asset:} \quad a_1 = S_2, \quad a_2 = -S_2 \tag{11}$$

The aggregate output and consumption of each agent are:

$$Y = A_1(S_1 + S_2),$$
$$c_1 = A_1(1 - f_1)(S_1 + S_2), \quad c_2 = f_1 A_1(S_1 + S_2)$$

3. *If* $f_1 \geq \frac{S_2}{S_1+S_2}$*, then the borrowing constraint is not binding, and there exists an equilibrium characterized by:*

$$\textit{Interest rate:} \quad r = A_1 \tag{12}$$

$$\textit{Physical capital:} \quad k_1 = S_2 + S_1, \quad k_2 = 0 \tag{13}$$

$$\textit{Financial asset:} \quad a_1 = S_2; \quad a_2 = -S_2 \tag{14}$$

The aggregate output and consumption of each agent are:

$$Y = A_1(S_1 + S_2), \tag{15}$$

$$c_1 = A_1 S_1, \quad c_2 = A_1 S_2. \tag{16}$$

Proof. See Appendix 6.4 □

Let us denote:

$$f_1^* \equiv \frac{A_2}{A_1}\frac{S_2}{S_1+S_2}, \quad f_1^{**} \equiv \frac{S_2}{S_1+S_2} \tag{17}$$

Comments. In any case, agent 1 (the agent with higher productivity) borrows, and agent 2 (the one with lower productivity) lends in the international financial market. Therefore borrowing constraint f_2 imposed on agent 2 (the lender) is irrelevant to the equilibrium outcomes.

Based on Proposition 3, those outcomes are given as follows.

$$r = \begin{cases} A_2 & \text{if} \quad f_1 \leq f_1^* \\ f_1 A_1\Big(1 + \frac{S_1}{S_2}\Big) & \text{if} \quad f_1^* < f_1 < f_1^{**} \\ A_1 & \text{if} \quad f_1 \geq f_1^{**} \end{cases}$$

$$c_1 = \begin{cases} A_1 S_1 \frac{A_2 - f_1 A_2}{A_2 - f_1 A_1} & \text{if} \quad f_1 \leq f_1^* \\ A_1(1 - f_1)(S_1 + S_2), & \text{if} \quad f_1^* < f_1 < f_1^{**} \\ A_1 S_1 & \text{if} \quad f_1 \geq f_1^{**} \end{cases}, \quad c_2 = \begin{cases} A_2 S_2 & \text{if} \quad f_1 \leq f_1^* \\ f_1 A_1(S_1 + S_2) & \text{if} \quad f_1^* < f_1 < f_1^{**} \\ A_1 S_2 & \text{if} \quad f_1 \geq f_1^{**} \end{cases}$$

$$Y = \begin{cases} A_2 S_2 + A_1 S_1 \frac{A_2 - f_1 A_2}{A_2 - f_1 A_1} & \text{if} \quad f_1 \leq f_1^* \\ A_1(S_1 + S_2), & \text{if} \quad f_1^* < f_1. \end{cases}$$

Corollary 2.

1. *If the collateral constraint imposed on the borrower is sufficiently strict, i.e,* $f_1 \leq f_1^*$*, the lender's consumption is independent of* f_1*. For higher values of* f_1*, i.e,* $f_1^* < f_1 < f_1^{**}$*, the lender's consumption is increasing in* f_1*.*

2. *If the collateral constraint imposed on the borrower is sufficiently strict, i.e,* $f_1 \leq f_1^*$*, the borrower's consumption is an increasing function of* f_1*. For higher values of* f_1*, i.e,* $f_1^* < f_1 < f_1^{**}$*, the borrower's consumption is decreasing in* f_1*.*

3. *If the collateral constraint imposed on the borrower is sufficiently loose, i.e,* $f_1 \leq f_1^*$*, the the aggregate output is increasing in* f_1*. For higher values of* f_1*, i.e,* $f_1^* < f_1 < f_1^{**}$*, the aggregate output is independent of* f_1*.*

Following Obiols-Homs (2011), we decompose the impact of f_1 on agents' consumption into "quantity effect" and "price effect". In particular, the quantity effect implies that loosening the collateral constraint imposed on the borrower (by increasing f_1) has a positive impact on the borrower whenever the borrowing constraint binds since it allows the borrower to increase his capital use for production. The "price effect" refers to the change of interest rate when the collateral constraint is loosen. The lender is only affected by the "price effects". The borrower, however, is affected by both the "quantity effect" and the "price effect".

When the collateral constraint is sufficiently loose ($f_1 \leq f_1^*$), interest rate is constant, therefore the price effect is zero. Consequently, the lender is unaffected by an increase of f_1, whereas, the "quantity effect" leads to a gain for the borrower. With higher values of f_1, ($f_1^* < f_1 < f_1^{**}$), the lender lends all his saving, i.e the supply of debts is given, an increase in the borrowing limit triggers higher demand for debts, then lifts the equilibrium interest rate. The "price effect" thus increases the lender's consumption and harms the borrower's. In this case, the "price effect" dominates the "quantity effect", which leads to a reduction in the borrower's consumption.

Corollary 3.

1. *When the borrowing constraint of the borrower binds, the interest rate is smaller than that in case without borrowing constraints.*

2. *When the borrowing constraint of the borrower binds, the the lender's consumption is smaller than that in the case without borrowing constraints.*

3. *When the borrowing constraint of the borrower binds, his consumption is greater than that in the case without borrowing constraints.*

4. *When the borrowing constraint of the borrower binds, the aggregate output in the case under borrowing constraint is smaller than or equal to that in the case without borrowing constraint. The equality occurs with higher value of* f_1 *(*$f_1 > f_1^*$*).*

Hence, we find similar conclusions with Obiols-Homs (2011). The lender always benefits from the increase in the borrowing limit, whereas, the borrower may lose if the initial borrowing limit is sufficiently large. In other words, the borrower's consumption follows an inverted U-shape as a function of the borrowing limit. As regards the aggregate welfare, the impact of loosening tight borrowing limits on it depends on how the aggregate welfare is measured, which we will discuss formally.

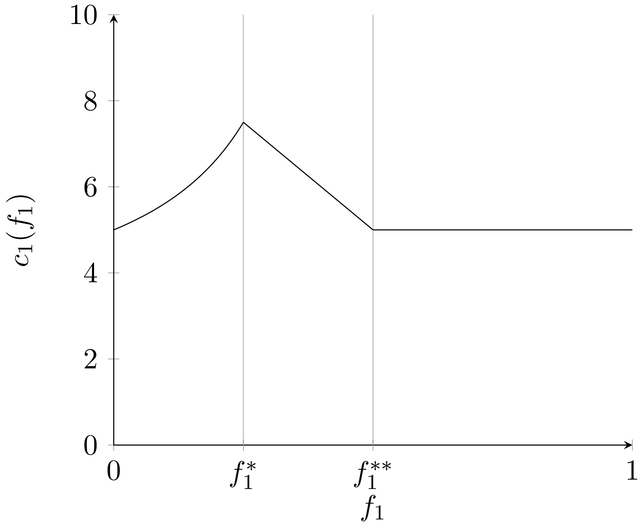

Figure 1: *Impact of f_1 on the borrower's consumption.* Parameters in function $c_1(f_1)$ are $A_1 = 1; A_2 = 0.5; S_1 = 5; S_2 = 5$

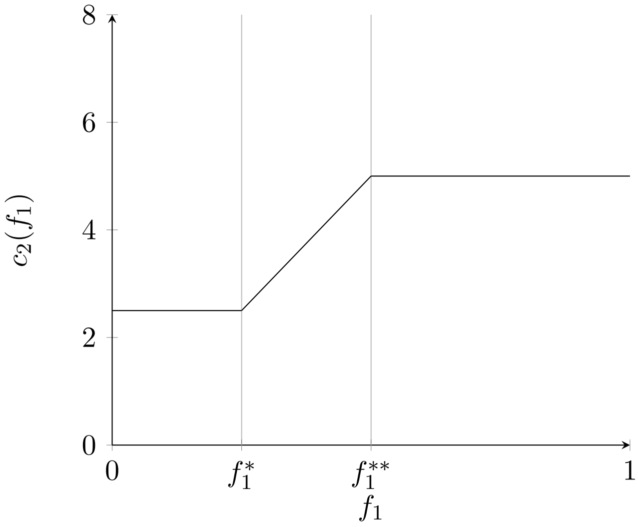

Figure 2: *Impact of f_1 on the lender's consumption.* Parameters in function $c_2(f_1)$ are $A_1 = 1; A_2 = 0.5; S_1 = 5; S_2 = 5$

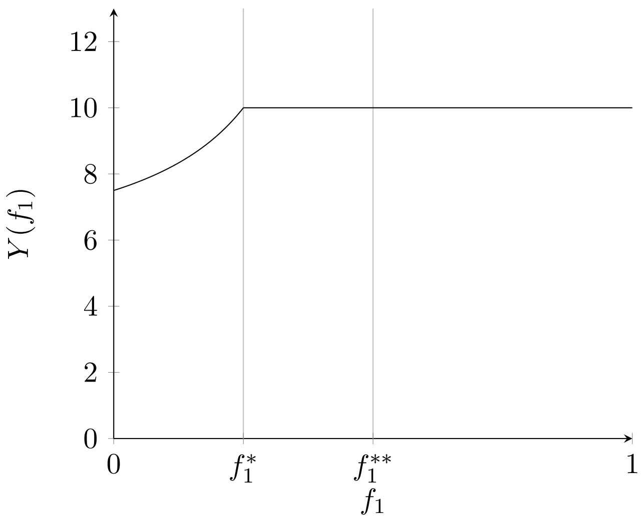

Figure 3: *Impact of f_1 on the aggregate output* Parameters in function $Y(f_1)$ are $A_1 = 1; A_2 = 0.5; S_1 = 5; S_2 = 5$

3.1.1 Aggregate welfare with collateral requirement: the linear technology case

Let us define the aggregate or social welfare (depending on $f \equiv (f_i)_{i=1,2}$) by:

$$\mathcal{W}(f) \equiv \sum_{i=1}^{2} \gamma_i u_i(c_i) \tag{18}$$

where u_i is the utility function of agent i, γ_i is weight assigned to agent i's utility.

Proposition 4. *Under Assumption 1 and 2, the aggregate welfare depends on the pledgeable fraction associated with the collateral constraint of the borrower, f_1 when this constraint binds ($f < f_1^{**}$). The impact of f_1 on the aggregate welfare depends on how it is aggregated and how the utility functions are defined.*

1. *If $u_i(c_i) = c_i$ for any i, then:*
   - *If $\gamma_2 = \gamma_1$, $\mathcal{W}(f_1, f_2)$ is increasing in f_1.*
   - *If $\gamma_2 > \gamma_1$, $\mathcal{W}(f_1, f_2)$ is increasing in f_1.*
   - *If $\gamma_1 > \gamma_2$, $\mathcal{W}(f_1, f_2)$ is increasing in f_1 if the initial value of f_1 for $f_1 \in [0, f_1^*]$, it is decreasing in f_1 for $f_1 \in (f_1^*, f_1^{**})$.*
2. *If $u_i(c_i) = ln(c_i)$ for any i, then*
   - *If $f_1^{**} \leq \gamma_2$, then $\mathcal{W}(f_1, f_2)$ is increasing in f_1.*
   - *If $f_1^* < \gamma_2 < f_1^{**}$, then $\mathcal{W}(f_1, f_2)$ is increasing in f_1 for $f_1 \in [0, \gamma_2]$ and decreasing in f_1 for $f_1 \in (\gamma_2, f_1^{**})$.*
   - *If $f_1^* \leq \gamma_2$, then $\mathcal{W}(f_1, f_2)$ is increasing in f_1 for $f_1 \in [0, f_1^*]$ and decreasing in f_1 for $f_1 \in \left(f_1^*, f_1^{**}\right)$.*
3. *If $u_i(c_i) = \dfrac{c_i^{1-\sigma}}{1-\sigma}$, then*
   - *If $f_i^{**} \leq \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}$, then $\mathcal{W}(f_1, f_2)$ is increasing in f_1.*
   - *If $f_1^* < \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}} < f_i^{**}$, then $\mathcal{W}(f_1, f_2)$ is increasing in f_1 for $f_1 \in \left[0, \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}\right]$ and decreasing in f_1 for $f_1 \in \left(\dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}, f_i^{**}\right)$.*
   - *If $f_1^* \geq \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}$, then $\mathcal{W}(f_1, f_2)$ is increasing in f_1 for $f_1 \in (0, f_1^*)$ and is decreasing in f_1 for $f_1 \in (f_1^*, f_1^{**})$*

Proof. See Appendix 6.5 □

Based on the correlation between collateral constraint imposed on the borrower and the aggregate welfare specified in Proposition 4, the optimal pledgeable fraction associated with this collateral constraint, denoted as $\hat{f}_1$, are characterized in the following result.

Proposition 5.

1. *If $u_i(c_i) = (c_i)$ for any i, then:*

$$\hat{f}_1 = \begin{cases} f_1^* & \text{if} \quad \gamma_1 > \gamma_2 \\ \text{any } f_1 \ \ s.t\ f_1 \geq f_1^{**} & \text{if} \quad \gamma_2 \geq \gamma_1 \end{cases}$$

2. *If $u_i(c_i) = ln(c_i)$ for any i, then:*

$$\hat{f}_1 = \begin{cases} f_1^* & \text{if} \quad \gamma_2 \leq f_1^* \\ \gamma_2 & \text{if} \quad f_1^* < \gamma_2 < f_1^{**} \\ \text{any } f_1 \ \ s.t\ f_1 \geq f_1^{**} & \text{if} \quad f_1^{**} \leq \gamma_2 \end{cases}$$

3. $u_i(c_i) = \dfrac{c_i^{1-\sigma}}{1-\sigma}$, *then:*

$$\hat{f}_1 = \begin{cases} f_1^* & \text{if} \quad f_1^* \geq \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}} \\ \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}} & \text{if} \quad f_1^* < \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}} < f_1^{**} \\ \text{any } f_1 \ \ s.t\ f_1 \geq f_1^{**} & \text{if} \quad f_1^{**} \leq \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}} \end{cases}$$

Firstly, we see that the higher weight put on the lender's consumption, the higher f_1, or the looser the collateral requirement should be.

We also find that the optimal collateral constraint is either the one from which the borrowing constraint is no longer binding or the point from which the lender's gain no longer compensate fully the borrower's loss as the borrowing limit increases.

When the weight assigned to the lender's consumption (γ_2) is sufficiently high, in the sense that $\gamma_2 \geq \gamma_1$ if the utility of an agent is simply his consumption, or $\gamma_2 \geq f_1^{**}$ if the utility of an agent is a logarithm function, or $\gamma_2 \geq \left(\frac{1-f_1^{**}}{f_1^{**}}\right)^{\sigma}\gamma_1$ if the utility of an agent is a CES function, which makes the gain of the lender more than compensate the loss of the borrower when the borrowing limit increases, and the former occurs. Hence, the optimal borrowing limit in this case is the maximum possible (the one at which the borrowing constraints start being not binding).

When the weight assigned to the borrower's consumption is sufficiently low, the gain of the lender could not fully compensate the loss of the borrower when the borrowing limit increases, and the latter occurs. Hence, the optimal limit in this case must be lower than the case with higher weight for the lender's consumption.

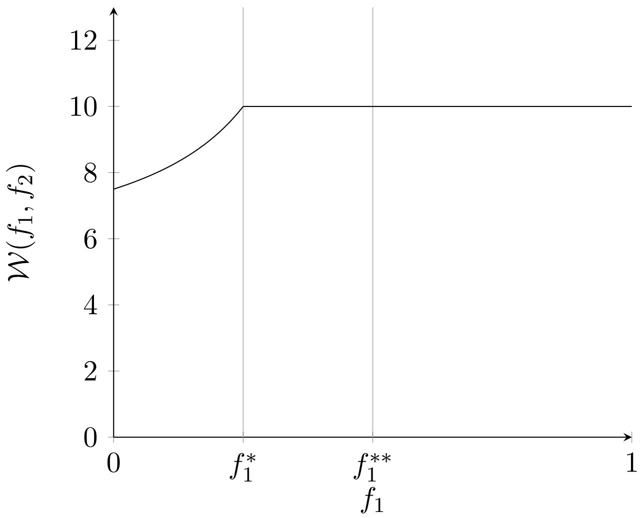

Figure 4: *Aggregate welfare measured by unweighted sum over agents' consumption.* Parameters used in function $\mathcal{W}(f_1, f_2)$ are $A_1 = 1; A_2 = 0.5; S_1 = S_2 = 5$

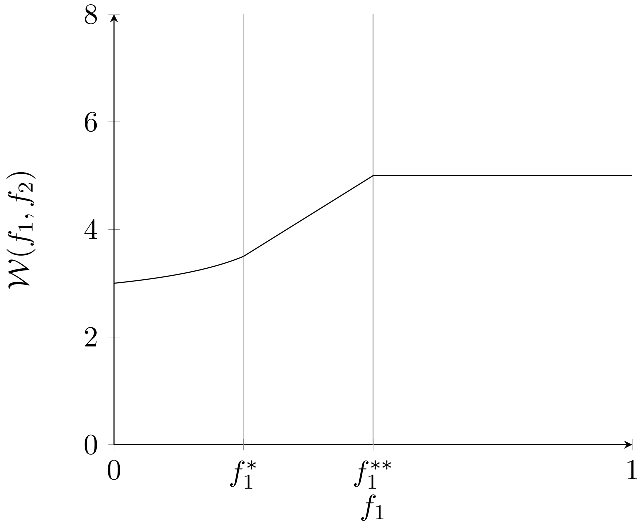

Figure 5: *Aggregate welfare measured by weighted sum over agents' consumption with higher weight for lender's consumption).* Parameters used in function $\mathcal{W}(f_1, f_2)$ are $A_1 = 1; A_2 = 0.5; S_1 = S_2 = 5, \gamma_1 = 0.2, \gamma_2 = 0.8$

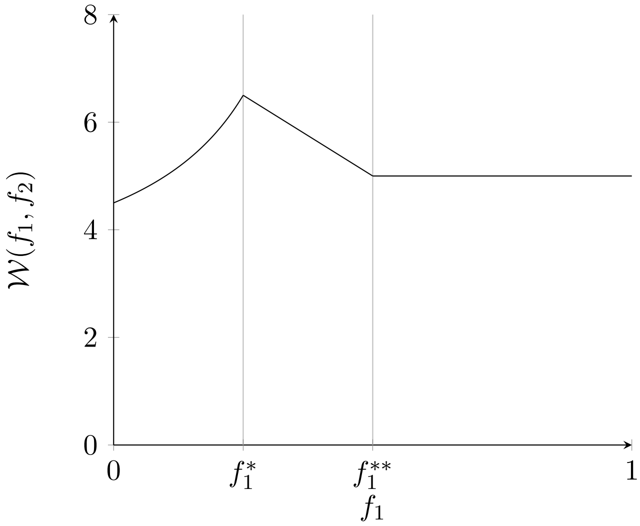

Figure 6: *Aggregate welfare measured by weighted sum over agents' consumption with higher weight for borrower's consumption).* Parameters used in function $\mathcal{W}(f_1, f_2)$ are$A_1 = 1; A_2 = 0.5; S_1 = S_2 = 5, \gamma_1 = 0.8, \gamma_2 = 0.2$

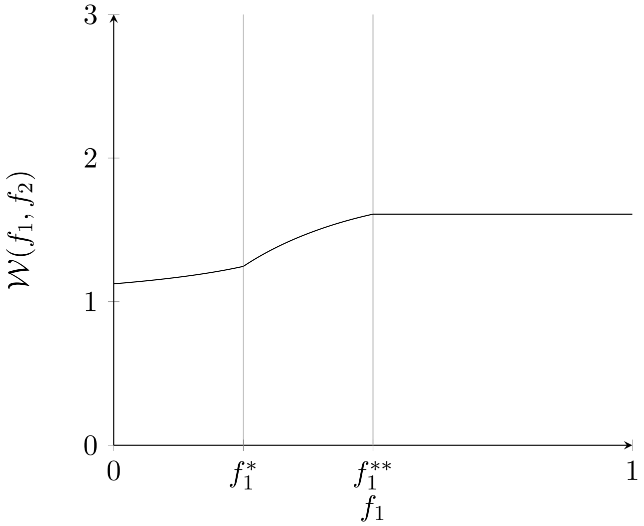

Figure 7: *Aggregate welfare measured by weighted sum over agents' logarithm utility functions.* Parameters used in function $\mathcal{W}(f_1, f_2)$ are $A_1 = 1; A_2 = 0.5; S_1 = S_2 = 5; \gamma_1 = 0.3, \gamma_2 = 0.7$

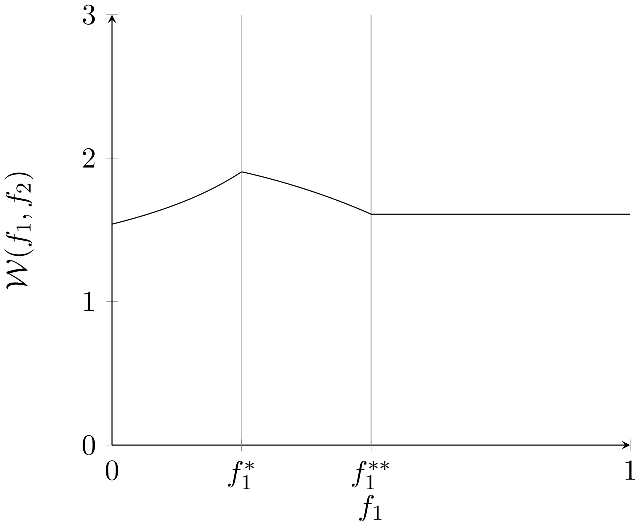

Figure 8: *Aggregate welfare measured by weighted sum over agents' logarithm utility functions.* Parameters used in function $\mathcal{W}(f_1, f_2)$ are $A_1 = 1; A_2 = 0.5; S_1 = S_2 = 5; \gamma_1 = 0.9, \gamma_2 = 0.1$

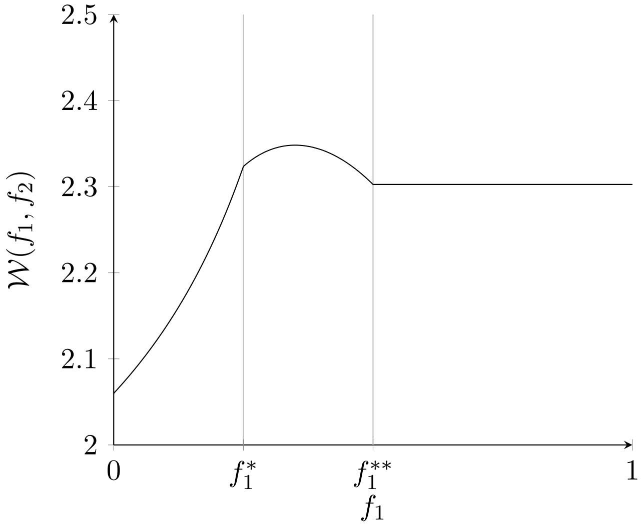

Figure 9: *Aggregate welfare measured by weighted sum over agents' logarithm utility functions.*Parameters used in function $\mathcal{W}(f_1, f_2)$ are $A_1 = 1; A_2 = 0.5; S_1 = S_2 = 10; \gamma_1 = 0.65, \gamma_2 = 0.35$

3.2 Cobb-Douglas technology

With $F_i(k) = A_i k^\alpha$, the problem (P_i) becomes:

$$(P_i'^C) \qquad c_i = \max_{k_i, a_i} [A_i k_i^\alpha - r a_i] \tag{19}$$

$$\text{subject to}: \qquad 0 \leq k_i \leq S_i + a_i \tag{20}$$

$$r a_i \leq f_i A_i k_i^\alpha \tag{21}$$

Lemma 4. *The solutions for the maximization problem of agent i with Cobb-Glass technology and collateral constrains are described in the following cases:*

1. *If $\Big(\frac{r}{\alpha A_i S_i^{\alpha-1}}\Big)^{\frac{1}{1-\alpha}} + \frac{f_i}{\alpha} \leq 1$, then the borrowing constraint is binding and agent i is a borrower, with:*

$$k_i^{1-\alpha} - \frac{S_i}{k_i^\alpha} = \frac{f_i A_i}{r}, \quad a_i = k_i - S_i$$

2. *If $\Big(\frac{r}{\alpha A_i S_i^{\alpha-1}}\Big)^{\frac{1}{1-\alpha}} + \frac{f_i}{\alpha} > 1$ then the borrowing constraint is not binding and*

$$k_i = \big(\frac{\alpha A_i}{r}\big)^{\frac{1}{1-\alpha}}, \quad a_i = k_i - S_i$$

Agent i lends if $r \geq \alpha A_i S_i^{\alpha-1}$, and borrows if $r < \alpha A_i S_i^{\alpha-1}$.

Proof. See Appendix 6.6 □

From Lemma 4, we see that when the borrowing limit is sufficiently hight, i.e, $f_i > \alpha$, then constraint is not binding, the capital use k_i of each agent depends on A and α.

The more interesting case is when the borrowing limit is low, i.e, $f_i < \alpha$, then the borrowing constraint is biding when $\alpha A_i S_i^{\alpha-1}$ is high enough. The intuition is that, the higher productive the agent is, the more he would like to borrow from the financial market. When the productivity is sufficiently high, the borrowing constraint binds.

We consider an agent with $\alpha A_i S_i^{\alpha-1} > r$. This condition is satisfied if the TFP A is high enough. According to the Lemma 4, the borrowing constraint is binding if and only if the borrowing limit is lower than a critical threshold, $\bar{f}$, which is given by:

$$\bar{f} = \alpha\Big(1 - \big(\frac{r}{\alpha A_i S_i^{\alpha-1}}\big)^{\frac{1}{1-\alpha}}\Big)$$

We see that:

$$\frac{\partial \bar{f}}{\partial r} < 0, \quad \frac{\partial \bar{f}}{\partial A_i} > 0, \quad \frac{\partial \bar{f}}{\partial S_i} < 0$$

Corollary 4. *Assume that $f_i > \alpha$ for any i. Then borrowing constraint of each agent does not bind and hence the equilibrium in the borrowing-constrained economy coincides with that in the economy without borrowing constraints.*

Let us denote

$$\bar{r}_i \equiv \alpha A_i S_i^{\alpha-1}\big(1 - \frac{f_i}{\alpha}\big)^{1-\alpha}, \quad \hat{r}_2 \equiv \alpha A_2 S_2^{\alpha-1}$$

Proposition 6. *Without loss of generality, we assume that $\bar{r}_1 > \bar{r}_2$.*

At equilibrium, agent i borrows if and only if his lowest marginal productivity in autarky ($\alpha A_i S_i^{\alpha-1}$) is greater than the other's, and:

1. *If*

$$\frac{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}}} \frac{S_1}{S_1 + S_2} + \frac{f_1}{\alpha} > 1 \tag{22}$$

then no borrowing constraint is binding and the equilibrium coincides with that in the economy without borrowing constraints.

2. *If*

$$\frac{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}}} \frac{S_1}{S_1 + S_2} + \frac{f_1}{\alpha} \leq 1 \tag{23}$$

then there exists a equilibrium determined by:

$$\text{Interest rate:} \quad S_2 - \left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}} = \frac{f_1 A_1}{r}\left[S_1 + S_2 - \left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}}\right]^{\alpha} \tag{24}$$

$$\text{Physical capital:} \quad k_1^{1-\alpha} - \frac{S_1}{k_1^{\alpha}} = \frac{f_1 A_1}{r}, \quad k_2 = \left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}} \tag{25}$$

$$\text{Financial asset:} \quad a_1 = k_1 - S_1, \quad a_2 = k_2 - S_2. \tag{26}$$

In this equilibrium, $r \in (\hat{r}_2, \bar{r})$ and the borrowing constraint of agent 1 is binding.

The aggregate output and consumption of each agent are:

$$Y = A_1\left[S_1 + S_2 - \left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}}\right]^{\alpha} + A_2\left(\frac{\alpha A_2}{r}\right)^{\frac{\alpha}{1-\alpha}},$$

$$c_1 = A_1\left[S_1 + S_2 - \left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}}\right]^{\alpha} - r\left[S_2 - \left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}}\right],$$

$$c_2 = A_2\left(\frac{\alpha A_2}{r}\right)^{\frac{\alpha}{1-\alpha}} + r\left(S_2 - \left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}}\right)$$

Proof. See Appendix 6.7 □

We see that with sufficiently high value of f_1, i.e, f_1 such that (22) holds, then the borrowing constraint is not binding, we achieve the same equilibrium outcomes as in the case without borrowing constraints, which are independent of the value of f_1. The impact of f_1 on the equilibrium outcomes when the borrowing constraint binds (f_1 is sufficiently low) are summarized in the following result.

Corollary 5. *At equilibrium where the borrowing constraint is binding, interest rate r is an increasing function of f_1.*

Proof. See Appendix 6.8 □

Since when the borrowing constraint of agent 1 binds, $r \in (\hat{r}_2, \bar{r})$, we thus have $f_1 \in (\underline{f}_1, \bar{f}_1)$ in this case, where $\underline{f}_1 = f_1(\hat{r}_2)$ and $\bar{f}_1 = f_1(\bar{r})$.

Corollary 6.

1. *When the borrowing constraint imposed on the borrower binds, the lender's consumption is an increasing function in* f_1
2. *When the borrowing constraint imposed on the borrower binds, there exist* $\tilde{f}_1$ *the borrower's consumption is an increasing function in* f_1 *for* $f_1 \in (f_1, \tilde{f}_1)$*, and is a decreasing function in* f_1 *for* $f_1 \in (\tilde{f}_1, \bar{f}_1$
3. *When the borrowing constraint imposed on the borrower binds, the aggregate output is an increasing function of* f_1*.*

Proof. See Appendix 6.9 □

We see that, the impact of pledgeable fraction associated with the collateral constraint of the borrower (f_1) on the lender's consumption and the aggregate output at equilibrium are in the same direction as in the case where agents take interest rate as exogenous. However, while loosening borrowing constraint imposed on the borrower always leads to a higher consumption for the borrower, it may reduce the borrower's consumption at equilibrium when r is determined by the market clearing condition. The intuition is that, when r is given by the market clearing condition, it will increase when f_1 grows, and hence reduce the borrower's consumption.

Corollary 7.

1. *Interest rate under collateral constraints is smaller than or equal to that in an unconstrained economy.*
2. *The lender's consumption under collateral constraints is smaller than or equal to that in an unconstrained economy.*
3. *There exists* $\tilde{f}_1 < \bar{f}_1$ *such that the borrower's consumption under collateral constraints is greater than or equal to that in an unconstrained economy at least with* $f_1 \in (\tilde{f}_1 < \bar{f}_1)$
4. *The aggregate output under collateral constraints is smaller than or equal to that in an unconstrained economy.*

Proof. See Appendix 6.10 □

The following figures illustrate the impacts of an increase in the borrowing limit of the borrower on consumption of the borrower, the lender, and the aggregate welfare, in a particular and simple case, when $\alpha = \frac{1}{2}$. Note that $\bar{f}_1$ in the graph denote the point from which the collateral constraint of the borrower start binding.

With $\alpha = \frac{1}{2}$, one can derive that:

$$r = \frac{A_2}{2} \frac{\Big[2\big(\frac{2f_1A_1}{A_2}\big)^2 + 2\Big]^{\frac{1}{2}}}{\Big[2S_2 + \big(\frac{2f_1A_1}{A_2}\big)^2(S_1+S_2) - \sqrt{\big(\frac{2f_1A_1}{A_2}\big)^4(S_1+S_2)^2 + 4S_1S_2\big(\frac{2f_1A_1}{A_2}\big)^2}\Big]^{\frac{1}{2}}}$$

$$c_1(r) = A_1\Big[S_1 + S_2 - \big(\frac{A_2}{2r}\big)^2\Big]^{\frac{1}{2}} - r\Big[S_2 - \big(\frac{A_2}{2r}\big)^2\Big], \quad c_2(r) = \frac{(A_2)^2}{2r} + r\Big[S_2 - \big(\frac{A_2}{2r}\big)^2\Big]$$

$$Y(r) = A_1\Big[S_1 + S_2 - \big(\frac{A_2}{2r}\big)^2\Big]^{\frac{1}{2}} + \frac{(A_2)^2}{2r}$$

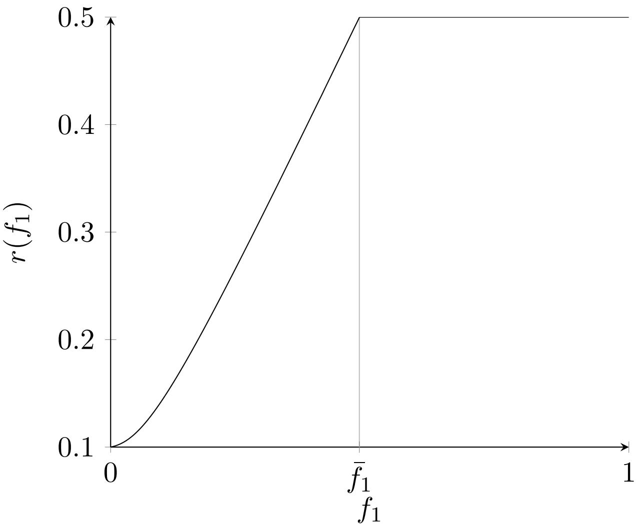

Figure 10: *Impact of f_1 on interest rate.* Parameters in function $r(f_1)$ are $A_1 = 5; A_2 = 1; S_1 = 1; S_2 = 25; \alpha = 0.5$

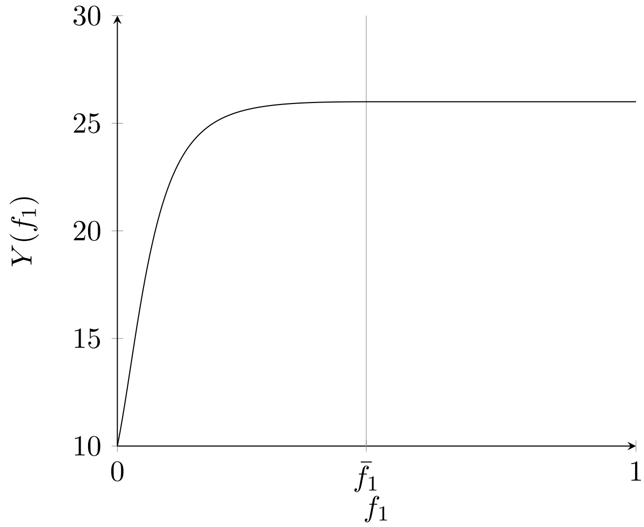

Figure 11: *Impact of f_1 on the aggregate output.*Parameters in function $Y(f_1)$ are $A_1 = 5; A_2 = 1; S_1 = 1; S_2 = 25; \alpha = 0.5$

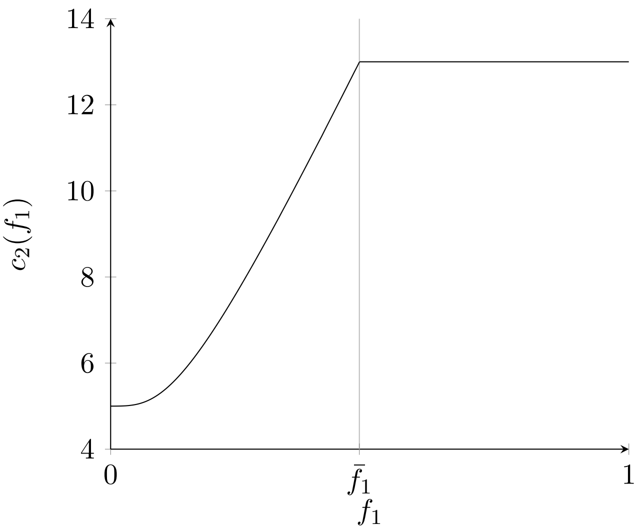

Figure 12: *Impact of f_1 on the lender's consumption* Parameters in function $c_2(f_1)$ are $A_1 = 5; A_2 = 1; S_1 = 1; S_2 = 25; \alpha = 0.5$

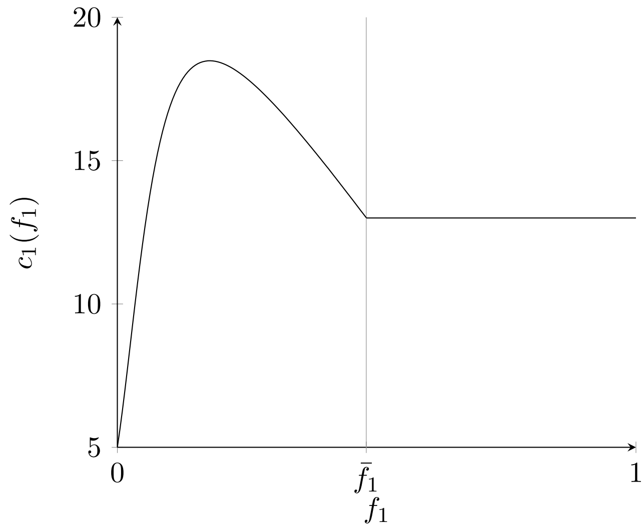

Figure 13: *Impact of f_1 on the borrower's consumption* Parameters in function $c_1(f_1)$ are $A_1 = 5; A_2 = 1; S_1 = 1; S_2 = 25; \alpha = 0.5$

4 A three-agent economy with collateral constraints

We now extend the framework of the subsection 3.1, assuming that there are three countries: H, F, G with linear technology and borrowing constraints.

On one hand, there must exists at least one lender among these countries at equilibrium.

According to Lemma 3, the country i lends his consumption if $A_i \leq r$. Hence, at equilibrium $r \geq Min\{A_H, A_F, A_G\}$.

On the other hand, there must also exists one borrower among countries at equilibrium. According to Lemma 3, the country i borrows from the financial market if $A_i \geq r$. Hence, at equilibrium, $r \leq Max\{A_H, A_F, A_G\}$.

In addition, with the same reasoning as we discussed in the subsection 3.1, we see that at equilibrium, $r > f_i A_i \quad$ for all i.

Assumption 3. $A_H < A_F < A_G$

Under assumption 3, we see that at equilibrium:

$$A_H \leq r \leq A_G$$

Each country i solves the problem $P_i'^L$, similar to the one discussed the subsection 3.1.

$$(P_i'^L) \qquad c_i = \max_{k_i, a_i} [A_i k_i - r a_i] \tag{27}$$

$$\text{subject to :} \qquad 0 \leq k_i \leq S_i + a_i \tag{28}$$

$$r a_i \leq f_i A_i k_i \tag{29}$$

Based on Lemma 3, we can characterize all kinds of equilibria.

Proposition 7.

1. *If* $\quad S_H \leq \dfrac{f_G}{1 - f_G} S_G - S_F,$

   there exists a unique equilibrium under which countries H and F lend, and country G borrows from the international financial market, with:

   Interest rate: $\quad r = A_F$
   Capital allocation: $\quad k_H = 0, \quad k_F = 0, \quad k_G = S_H + S_F + S_G$
   Financial asset: $\quad a_H = -S_H, \quad a_F = -S_F, \quad a_G = S_H + S_F$

   The aggregate output and consumption of each country are:

   $$Y = A_F(S_H + S_F + S_G)$$
   $$c_i = A_G S_i \quad \forall i \in \{H, F, G\}$$

2. *If* $\dfrac{f_G}{1 - f_G} S_G - S_F < S_H \leq \dfrac{f_G A_G}{A_F - f_G A_G} - S_F,$

   then there exists a unique equilibrium under which countries H and F lend, and country G borrows in the international financial market, with:

   Interest rate: $\quad r \in (A_F, A_G) \quad$ *and* $\quad r = f_G A_G \left(1 + \dfrac{S_G}{S_H + S_F}\right)$
   Capital allocation: $\quad k_H = 0, \quad k_F = 0, \quad k_G = \dfrac{r}{r - f_G A_G} S_G$
   Financial assets: $\quad a_H = -S_H, \quad a_F = -S_F, \quad a_G = \dfrac{f_G A_G}{r - f_G A_G} S_G$

The aggregate output and consumption of each country:

$$Y = \frac{r}{r - f_G A_G} A_G S_G$$
$$c_i = r S_i \quad \forall i \in \{H, F\}, \qquad c_G = \frac{r(1 - f_G)}{r - f_G A_G} A_G S_G$$

3. *If* $\dfrac{f_G A_G}{A_F - f_G A_G} S_G - S_F < S_H \leq \dfrac{f_G A_G}{A_F - f_G A_G} S_G,$

there exists a unique equilibrium, under which countries H and F lend, country G borrows in the international financial market, with:

Interest rate: $r = A_F$

Capital allocation: $k_H = 0, \quad k_G = \dfrac{A_F}{A_F - f_G A_G}, \quad k_F = S_F + S_H - \dfrac{f_G A_G}{A_F - f_G A_G} S_G$

Financial assets: $a_H = -S_H, \quad a_G = \dfrac{f_G A_G}{A_F - f_G A_G} S_G, \quad a_F = S_H - \dfrac{f_G A_G}{A_F - f_G A_G}$

The aggregate output and consumption of each country are:

$$Y = \frac{1 - f_G}{A_F - f_G A_G} A_F A_G S_G + A_F(S_F + S_H)$$
$$c_H = A_F S_H, \quad c_F = A_F S_F, \quad c_G = \frac{(1 - f_G) A_F}{A_F - f_G A_G} A_G S_G$$

4. *If* $\dfrac{f_G A_G}{A_F - f_G A_G} S_G < S_H \leq \dfrac{f_G A_G}{A_F - f_G A_G} S_G + \dfrac{f_F}{1 - f_F} S_F,$

there exists a unique equilibrium, under which country H lends, countries F and G borrow in the international financial market.

Interest rate: $r = A_F$

Capital allocation: $k_H = 0, \quad k_G = \dfrac{A_F}{A_F - f_G A_G}, \quad k_F = S_F + S_H - \dfrac{f_G A_G}{A_F - f_G A_G} S_G$

Financial assets: $a_H = -S_H, \quad a_G = \dfrac{f_G A_G}{A_F - f_G A_G} S_G, \quad a_F = S_H - \dfrac{f_G A_G}{A_F - f_G A_G}$

The aggregate output and consumption of each country are:

$$Y = \frac{1 - f_G}{A_F - f_G A_G} A_F A_G S_G + A_F(S_F + S_H)$$
$$c_H = A_F S_H, \quad c_F = A_F S_F, \quad c_G = \frac{(1 - f_G) A_F}{A_F - f_G A_G} A_G S_G$$

5. *If* $\dfrac{f_G A_G}{A_F - f_G A_G} S_G + \dfrac{f_F}{1 - f_F} S_F < S_H < \dfrac{f_G A_G}{A_H - f_G A_G} S_G + \dfrac{f_F A_F}{A_H - f_F A_F} S_F,$

there exists a unique equilibrium under which country H lends, countries F and G borrow from the international financial market, with:

$$\text{Interest rate: } \quad r \in (A_H, A_F); \quad \frac{f_F A_F}{r - f_F A_F} S_F + \frac{f_G A_G}{r - f_G A_G} S_G = S_H$$
$$\text{Capital allocation: } \quad k_H = 0, \quad k_F = \frac{r}{r - f_F A_F} S_F, \quad k_G = \frac{r}{r - f_G A_G} S_G$$
$$\text{Financial assets: } \quad a_H = -S_H, \quad a_F = \frac{f_F A_F}{r - f_F A_F} S_F, \quad a_G = \frac{f_G A_G}{r - f_G A_G} S_G$$

The aggregate output and consumption of each country:

$$Y = \frac{r}{r - f_F A_F} A_F S_F + \frac{r}{r - f_G A_G} A_G S_G$$
$$c_H = r S_H, \quad c_F = \frac{r(1 - f_F)}{r - f_F A_F} A_F S_F, \quad c_G = \frac{r(1 - f_G)}{r - f_G A_G} A_G S_G$$

6. *If* $S_H \geq \frac{f_G A_G}{A_H - f_G A_G} S_G + \frac{f_F A_F}{A_H - f_F A_F} S_F,$

   there exists a unique equilibrium where country H lends, countries F and G borrow in the international financial market, with:

$$\text{Interest rate: } \quad r = A_H$$
$$\text{Financial assets: } \quad a_H = -\frac{f_F A_F}{A_H - f_F A_F} S_F - \frac{f_G A_G}{A_H - f_G A_G} S_G, \quad a_i = \frac{f_i A_i}{A_H - f_i A_i} S_i, \quad \forall i \in \{F, G\}$$
$$\text{Capital allocation: } \quad k_H = S_H - \frac{f_F A_F}{A_H - f_F A_F} S_F - \frac{f_G A_G}{A_H - f_G A_G} S_G,$$
$$k_i = \frac{A_H}{A_H - f_i A_i} S_i, \quad \forall i \in \{F, G\}$$

The aggregate output and consumption of each country:

$$Y = a_H S_H + \frac{1 - f_F}{A_H - f_F A_F} A_H A_F S_F + \frac{1 - f_G}{A_G - f_G A_G} A_H A_G S_G$$
$$c_H = A_H S_H, \quad c_i = \frac{(1 - f_i)}{A_H - f_i A_i} A_H A_i S_i \quad \forall i \in \{F, G\}$$

Proof. See Appendix 6.11. □

Comments.

1. Country H (the least productive country) always lends, and country G (the most productive country) always borrows.

2. Country F (the country with intermediate productivity) lends in some cases and borrows in some other cases, depending on the values of parameters.

   Let us denote $\bar{A}_F \equiv f_G A_G \Big(1 + \frac{S_G}{S_H}\Big)$

   We find that country F borrows from the financial market if the productivity of this country is greater than the threshold $\bar{A}_F$, and lends to the financial market otherwise. Furthermore, the higher $\bar{A}_F$ is, the more likely that country F is lender.

Corollary 8. *Under assumption 3:*

$$\frac{\partial \bar{A}_F}{\partial f_G} > 0, \quad \frac{\partial \bar{A}_F}{\partial A_G} > 0, \quad \frac{\partial \bar{A}_F}{\partial S_G} > 0, \quad \frac{\partial \bar{A}_F}{\partial S_H} < 0$$

Thus, the more financially developed country G becomes, the more likely that country F is a lender. This can be explained by the well-known phenomenon in the literature, capital tends to go from the less financially developed country to the more financially developed. Accordingly, when the financial market of country G develops, the more capital flows to this country, and country F, hence, is more likely to lend country G.

In addition, the more productive country G is, the more likely country F is a lender. It is because, the more productive country G is, the higher amount for capital it demands for production, the higher possibility that country F is a lender to fill the country G's increased demand in the financial market.

Finally, the higher saving that country H possesses, compared to country G's saving, the more likely country F is a borrower. The explanation is quite straightforward. An increase in the saving of county H relatively to country G's saving, i.e, an increase in the supply of debts in the financial market, which requires a corresponding growth in the demand side of the financial market. Hence, country F is more likely to be a borrower to contribute to the rise in the demand side.

5 Conclusion

We have constructed a tractable general equilibrium model with heterogeneous agents and financial market imperfections which induces interesting results. We managed to show that loosening tight collateral constraints is not always good to agents participating the financial market and the aggregate welfare. As a result, the optimal collateral requirement, which maximizes the social welfare, could be a binding constraint. We also characterized the capital flows under the asymmetry of financial development among countries. In the simple two-agent (two-country) framework, we showed that the direction of capital flows is entirely dictated by the difference in marginal productivity. Collateral constraints affect, if any, only the magnitude of capital flows. More interestingly, in a 3-agent (3-country) framework, we found that that while the most productive country always borrows, the least productive always lends, whether the country with intermediate level of productivity has capital inflows or outflows depends on its productivity, financial development and productivity of the most productive country, and savings of the other countries.

Our paper can be extended by several ways. First, one can allow agents' savings to be endogenous to solve our research questions in a dynamic framework where agents could adjust their savings when the collateral constraints are changed. Another line of research is to introduce market incompleteness and uncertainty, with two states of nature in the second period, for instance, and then investigate the impact of this type of financial friction on welfare and capital flows.

6 Appendix

6.1 Proof of Lemma 1.

It is obvious that at optimum: $k_i = S_i + a_i \geq 0 \Leftrightarrow a_i \geq -S_i$, the maximization problem of agent i thus becomes:

$$(P_i^L) \qquad c_i = \max_{a_i} [(A_i - r)a_i + A_i S_i]$$
$$\text{subject to :} \qquad 0 \leq a_i \geq S_i$$

Then, the solution of (P_i^L) are described in three cases as follows.

1. If $A_i > r$, then $a_i = +\infty$, $k_i = +\infty$
2. If $A_i < r$, then $a_i = -S_i$, $k_i = 0$
3. If $A_i = r$, then $c_i = AS_i$ for any pairs (a_i, k_i) that satisfies $a_i \geq -S_i$ and $k_i = a_i + S_i$.

□

6.2 Proof of Lemma 2.

Since $F_i'(0) = +\infty$, then at optimum, $k_i > 0$, the Lagrange function for (P_i^C) is:

$$L = [A_i k_i^{\alpha} - r a_i] + \lambda_i (S_i + a_i - k_i)$$

We see that (a_i, k_i) is a solution of the agent i's problem if and only if there exists λ_i such that:

$$[k_i] : \alpha A_i k_i^{\alpha-1} = \lambda_i$$
$$[a_i] : r = \lambda_i$$
$$\lambda_i \geq 0, k_i - S_i - a_i \leq 0, \lambda_i (k_i - S_i - a_i) = 0$$

It is easy to see that at optimum $k_i = a_i + S_i$, which implies that $\lambda_i \geq 0$. We thus can derive that at optimum:

$$\alpha A_i k_i^{\alpha-1} = r \Leftrightarrow k_i = \left(\frac{\alpha A_i}{r}\right)^{\frac{1}{1-\alpha}} \tag{30}$$

a_i is then given by $a_i = k_i - S_i = \left(\frac{\alpha A_i}{r}\right)^{\frac{1}{1-\alpha}}$ □

6.3 Proof of Lemma 3.

Lagrange function for $(P_i'^L)$ with linear technology and borrowing constraints:

$$L = A_i k_i - r a_i + \lambda_i k_i + \mu_i (s_i + a_i - k_i) + \zeta_i (f_i A_i k_i - r a_i)$$

It is easy to see that at optimum: $k_i = a_i + S_i$.

We see that (k_i, a_i) is a solution if and only if there exists (λ_i, ζ_i) such that the FOCs with respect to k_i and a_i are satisfied.

$$[k_i] : (1 + \zeta_i f_i) A_i = \mu_i - \lambda_i \tag{31}$$
$$[a_i] : (1 + \zeta_i) r = \mu_i \tag{32}$$
$$\zeta_i \geq 0, \quad \zeta_i (r a_i - f_i A_i k_i) = 0 \tag{33}$$
$$\lambda_i \geq 0, \quad \lambda_i k_i = 0 \tag{34}$$

Case 1: The borrowing constrain is binding: $ra_i = f_i A_i k_i$, hence $\zeta_i \geq 0$ and k_i, a_i is the solution of the following equations:

$$ra_i = f_i A_i k_i$$
$$k_i = S_i + a_i$$

We can compute the solution as follows

$$k_i = \frac{r}{r - f_i A_i} S_i, \quad a_i = \frac{f_i A_i}{r - f_i A_i} S_i$$

Therefore: $r > f_i A_i$, and since $k_i > 0$, we have $\lambda_i = 0$.

From 31 and 32, we thus have:

$$\frac{A_i}{r} = \frac{1 + \zeta_i}{1 + \zeta_i f_i} \geq 1$$
$$\Leftrightarrow A_i \geq r$$

So when $A_i \geq r > f_i A_i$, the borrowing is binding and the solution as we just computed in this case. We thus prove the second part of the Lemma 3.

Case 2: The borrowing is not binding: $ra_i < f_i A_i k_i$, hence $\zeta_i = 0$.

From (31) and (32), we have:

$$A_i = \mu_i - \lambda_i$$
$$r = \mu_i$$

- If $k_i > 0$, then $\lambda_i = 0$, which implies that $A_i = r$
- If $k_i = 0$, then $a_i = -S_i$, $\lambda \geq 0$ and $A_i \leq r$.

Thus, when $A_i = r$, any set (a_i, k_i) such that $S_i \leq a_i \leq f_i A_i$, and $k_i = a_i + S_i$ is the solution of the agent i's maximization problem. When $A_i \leq r$, then the solution is unique and given by: $k_i = 0$, then $a_i = -S_i$.

We hence proved the first and third part of the Lemma 3.

□

6.4 Proof of Proposition 3

From Lemma 3, we see that, at equilibrium $Min\{A_1, A_2\} \leq r \leq Max\{A_1, A_2\}$.

Under the assumption 1 ($A_2 < A_1$), we thus have at equilibrium, $A_2 \leq r \leq A_1$. There are three cases, each having a unique equilibrium described as follows.

1. If $A_2 = r < A_1$, based on Lemma 3, we see that there exists an equilibrium determined by:

$$k_1 = \frac{r}{r - f_1 A_1} S_1, \quad a_1 = \frac{f_1 A_1}{r - f_1 A_1} S_1$$
$$k_2 = S_2 - \frac{f_1 A_1}{r - f_1 A_1} S_1, \quad a_2 = -\frac{f_1 A_1}{r - f_1 A_1} S_1$$

In this case, agent 2 is the lender and agent 1 is the borrower. We see that $ra_1 = f_1 A_1 k_1$, i.e the borrowing constraint is satisfied. We need to check the non-negative condition on k_1, k_2. It is easy to see that $k_1 \geq 0$. Then one more condition left to be checked, that is $k_2 \geq 0$.

This condition is satisfied if:

$$S_2 - \frac{f_1 A_1}{r - f_1 A_1} S_1 \geq 0$$
$$\Leftrightarrow f_1 \leq \frac{A_2}{A_1} \frac{S_2}{S_1 + S_2}$$

2. If $A_2 < r < A_1$, based on Lemma 3, we see that there exists an equilibrium determined by:

$$k_1 = \frac{r}{r - f_1 A_1} S_1, \quad k_2 = 0$$
$$a_1 = \frac{f_1 A_1}{r - f_1 A_1} S_1, \quad a_2 = -S_2;$$

The equilibrium interest rate is given by:

$$\frac{f_1 A_1}{r - f_1 A_1} S_1 = S_2 \Leftrightarrow r = f_1 A_1 \Big(1 + \frac{S_1}{S_2}\Big)$$

We see that $r \in (A_2, A_1)$ if and only if:

$$A_2 < f_1 A_1 \Big(1 + \frac{S_1}{S_2}\Big) < A_1$$
$$\Leftrightarrow \frac{A_2}{A_1} \frac{S_2}{S_1 + S_2} < f_1 < \frac{S_2}{S_1 + S_2}$$

3. If $A_2 < r = A_1$, that there exists an equilibrium determined by:

$$k_2 = 0, \quad a_2 = -S_2$$
$$k_1 = S_1 + S_2, \quad a_1 = S_2$$

In this case, we need to verify the borrowing constraint of the borrwer (agent 1): $ra_1 \leq f_1 A_1 k_1$, or $a_1 \leq f_1(a_1 + S_1)$ This condition is satisfied when:

$$S_2 \leq f_1(S_1 + S_2) \Leftrightarrow f_1 \leq \frac{S_2}{S_1 + S_2}$$

□

6.5 Proof of Proposition 4

Let us recall that when $f_1 \geq f_1^{**}$, the borrowing constraint imposed on agent 1 does not bind, all the equilibrium' results coincide with the case without borrowing constraints, and the aggregate welfare does not depend on f_1. Therefore, we just investigate the impact of f_1 when the borrowing constraint of agent 1 binds, i.e, when $f_1 < f_1^{**}$.

1. $u_i(c_i) = c_i$ for any i.

$$\mathcal{W}(f_1, f_2) = \gamma_1 c_1 + \gamma_2 c_2 = \begin{cases} \gamma_1 A_1 S_1 \frac{A_2 - f_1 A_2}{A_2 - f_1 A_1} + \gamma_2 A_2 S_2 & \text{if} \quad f_1 \leq f_1^* \\ \gamma_1 A_1 (S_1 + S_2) + (\gamma_2 - \gamma_1) f_1 A_1 (S_1 + S_2) & \text{if} \quad f_1^* < f_1 < f_1^{**} \end{cases}$$

Hence,

- If $f_1 \leq f_1^*$, then $\mathcal{W}(f_1,f_2) = \gamma_1 A_1 S_1 \dfrac{A_2 - f_1 A_2}{A_2 - f_1 A_1} + \gamma_2 A_2 S_2$ is an increasing function in f_1.
- If $f_1^* < f_1 < f_1^{**}$, then $\mathcal{W}(f_1,f_2) = \gamma_1 A_1(S_1+S_2) + (\gamma_2 - \gamma_1) f_1 A_1 (S_1+S_2)$, which is increasing in f_1 if and only if $\gamma_1 \leq \gamma_2$, and decreasing in f_1 otherwise.

2. If $u_i(c_i) = ln(c_i)$ for any i.

$\mathcal{W}(f_1,f_2) = \gamma_1 ln(c_1) + \gamma_2 ln(c_2)$, or:

$$\mathcal{W}(f_1,f_2) = \begin{cases} \gamma_1 ln\Big(A_1 S_1 \dfrac{A_2 - f_1 A_2}{A_2 - f_1 A_1}\Big) + \gamma_2 \ln(A_2 S_2) & \text{if} \quad f_1 \leq f_1^* \\ \gamma_1 ln\big(A_1(1-f_1)(S_1+S_2)\big) + \gamma_2 ln\big(f_1 A_1 (S_1+S_2)\big) & \text{if} \quad f_1^* < f_1 < f_1^{**} \end{cases}$$

Hence,

- If $f_1 \leq f_1^*$, then $\mathcal{W}(f_1,f_2) = ln\Big(A_1 S_1 \dfrac{A_2 - f_1 A_2}{A_2 - f_1 A_1}\Big) + \ln(A_2 S_2)$, which is increasing in f_1
- If $f_1^* < f_1 < f_1^{**}$, then:

$$\begin{aligned} \mathcal{W}(f_1,f_2) &= \gamma_1 ln\big(A_1(1-f_1)(S_1+S_2)\big) + \gamma_2 ln\big(f_1 A_1(S_1+S_2)\big) \\ &= \gamma_1 ln(1-f_1) + \gamma_2 ln(f_1) + ln(A_1) + ln(S_1+S_2) \\ \mathcal{W}'_{f_1}(f_1,f_2) &= \frac{\gamma_2}{f_1} - \frac{\gamma_1}{(1-f_1)} = \frac{\gamma_2 - f_1}{f_1(1-f_1)} \end{aligned}$$

  Thus, $\mathcal{W}'_{f_1}(f_1,f_2) > 0$ if $f_1 < \gamma_2$ and $\mathcal{W}'_{f_1}(f_1,f_2) < 0$ if $f_1 > \gamma_2$.
  We consider following cases:
  Case 1: $f_1^{**} \leq \gamma_2$, or $\dfrac{S_2}{S_1+S_2} < \gamma_2$, then $\mathcal{W}(f_1,f_2)$ is an increasing function for all $f_1 \in (f_1^*, f_1^{**})$
  Case 2: $f_1^* \geq \gamma_2$, then $\mathcal{W}(f_1,f_2)$ is a decreasing function for all $f_1 \in (f_1^*, f_1^{**})$
  Case 3: $f_1^* < \gamma_2 < f_1^{**}$, then $\mathcal{W}(f_1,f_2)$ is an increasing function in (f_1^*, γ_2) and decreasing function in (γ_2, f_1^{**})

To sum up,

- If $f_1^{**} \leq \gamma_2$, then $\mathcal{W}(f_1,f_2)$ is increasing in f_1.
- If $f_1^* < \gamma_2 < f_1^{**}$, then $\mathcal{W}(f_1,f_2)$ is increasing in f_1 for $f_1 \in [0,\gamma_2]$ and decreasing in f_1 for $f_1 \in (\gamma_2, f_1^{**})$.
- If $f_1^* < \gamma_2$, then $\mathcal{W}(f_1,f_2)$ is increasing in f_1 for $f_1 \in [0, f_1^*]$ and decreasing in f_1 for $f_1 \in \big(f_1^*, f_1^{**}\big)$.

3. $u_i(c_i) = \dfrac{c_i^{1-\sigma}}{1-\sigma}$ for any i

$\mathcal{W}(f_1,f_2) = \gamma_1 \dfrac{c_1^{1-\sigma}}{1-\sigma} + \gamma_2 \dfrac{c_2^{1-\sigma}}{1-\sigma}$,or:

$$\mathcal{W}(f_1,f_2) = \begin{cases} \gamma_1 \dfrac{\Big(A_1 S_1 \dfrac{A_2 - f_1 A_2}{A_2 - f_1 A_1}\Big)^{1-\sigma}}{1-\sigma} + \gamma_2 \dfrac{(A_2 S_2)^{1-\sigma}}{1-\sigma} & \text{if } f_1 \leq f_1^* \\ = \gamma_1 \dfrac{\Big(A_1(1-f_1)(S_1+S_2)\Big)^{1-\sigma}}{1-\sigma} + \gamma_2 \dfrac{\Big(f_1 A_1(S_1+S_2)\Big)^{1-\sigma}}{1-\sigma} & \text{if } f_1^* < f_1 < f_1^{**} \end{cases}$$

Hence,

- If $f_1 \le f_1^*$, then $\mathcal{W}(f_1, f_2) = \gamma_1 \dfrac{\Big(A_1 S_1 \dfrac{A_2 - f_1 A_2}{A_2 - f_1 A_1}\Big)^{1-\sigma}}{1-\sigma} + \gamma_2 \dfrac{(A_2 S_2)^{1-\sigma}}{1-\sigma}$, which is increasing in f_1.
- If $f_1^* < f_1 < f_1^{**}$, then:

$$\mathcal{W}(f_1, f_2) = \gamma_1 \frac{\Big(A_1(1-f_1)(S_1+S_2)\Big)^{1-\sigma}}{1-\sigma} + \gamma_2 \frac{\Big(f_1 A_1 (S_1+S_2)\Big)^{1-\sigma}}{1-\sigma}$$
$$\mathcal{W}'_{f_1}(f_1, f_2) = [\gamma_2 f_1^{-\sigma} - \gamma_1 (1-f_1)^{-\sigma}]\big(A_1(S_1+S_2)\big)^{1-\sigma}$$

One can verify that:

$$\mathcal{W}'_{f_1}(f_1, f_2) > 0 \Leftrightarrow f_1 < \frac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}$$
$$\mathcal{W}'_{f_1}(f_1, f_2) < 0 \Leftrightarrow f_1 > \frac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}$$

We consider following cases:

Cases 1: $f_1^{**} < \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}$, then $\mathcal{W}(f_1, f_2)$ is an increasing function in f_1 for all $f_1 \in (f_1^*, f_1^{**})$.

Cases 2: $f_1^* > \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}$, then $\mathcal{W}(f_1, f_2)$ is a decreasing function in f_1 for all $f_1 \in (f_1^*, f_1^{**})$.

Cases 3: $f_1^* < \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}} < f_1^{**}$, then $\mathcal{W}(f_1, f_2)$ is increasing in f_1 for $f_1 \in \Big(f_1^*, \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}\Big)$ and decreasing in f_1 for $f_1 \in \Big(\dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}, f_1^{**}\Big)$.

To sum up,

- If $f_i^{**} \le \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}$, then $\mathcal{W}(f_1, f_2)$ is increasing in f_1.
- If $f_1^* < \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}} < f_i^{**}$, then $\mathcal{W}(f_1, f_2)$ is increasing in f_1 for $f_1 \in \Big[0, \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}\Big]$ and decreasing in f_1 for $f_1 \in \Big(\dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}, f_i^{**}\Big)$.
- If $f_1^* \ge \dfrac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}} + \gamma_2^{\frac{1}{\sigma}}}$, then $\mathcal{W}(f_1, f_2)$ is increasing in f_1 for $f_1 \in (0, f_1^*)$ and is decreasing in f_1 for $f_1 \in (f_1^*, f_1^{**})$

$\square$

6.6 Proof of Lemma 4

Assume that $F'(0) = 0$, then at optimum, $k_i > 0$
Lagrange function for $(P_i'^C)$ with Cobb-Douglas technology and borrowing constraint is given as:

$$L = A_i k_i^\alpha - r a_i + \lambda_i(s_i + a_i - k_i) + \mu_i(f_i A_i k_i^\alpha - r a_i)$$

It is easy to see that (k_i, a_i) is a solution if and only if there exists μ_i such that

$$[k_i] : (1 + \mu_i f_i)\alpha A_i k_i^{\alpha-1} = \lambda_i$$
$$[a_i] : (1 + \mu_i)r = \lambda_i$$
$$\mu_i \geq 0, \text{ and } \mu_i(f_i A_i k_i^\alpha - a_i) = 0.$$

These equations imply that:

$$\frac{\alpha A_i k_i^{\alpha-1}}{r} = \frac{1 + \mu_i}{1 + f_i \mu_i} \geq 1 \tag{35}$$

Case 1: The borrowing constraint is binding: $f_i A_i k_i^\alpha = r a_i$. In this case, (k_i, a_i) is the solutions of the following equations:

$$k_i^{1-\alpha} - \frac{S_i}{k_i^\alpha} = \frac{f_i A_i}{r} \tag{36}$$
$$a_i = k_i - S_i \tag{37}$$

We see that the left hand-side of the equation (36) is an increasing function in k_i. And, the left-hand side of (36) goes to $-\infty$ as k_i goes to 0. Hence, the equation (36) has a solution such that $0 < k_i \leq (\frac{\alpha A_i}{r})^{\frac{1}{1-\alpha}}$ if and only if:

$$\frac{\alpha A_i}{r} - S_i\Big(\frac{r}{\alpha A_i}\Big)^{\frac{\alpha}{1-\alpha}} \geq \frac{f_i A_i}{r}$$
$$\Leftrightarrow \quad \Big(\frac{r}{\alpha A_i S_i^{\alpha-1}}\Big)^{\frac{1}{1-\alpha}} \leq 1 - \frac{f_i}{\alpha}$$

From (36), we see that: $k_i - S_i = \frac{f_i A_i}{r} k_i^\alpha > 0$. Therefore, in this case, agent i is always a borrower.

We also see that $k_i \leq (\frac{\alpha A_i}{r})^{\frac{1}{1-\alpha}}$ implies that $\frac{\alpha A_i k_i^{\alpha-1}}{r} \geq 1$, and therefore $\mu_i \geq 0$.

Case 2: $f_i A_i k_i^\alpha > r a_i$. We see that $\mu_i = 0$, and hence $\frac{\alpha A_i k_i^{\alpha-1}}{r} = 1$, i.e, $k_i = (\frac{\alpha A_i}{r})^{\frac{1}{1-\alpha}}$. It remains to check that this value of k_i satisfies the condition: $f_i A_i k_i^\alpha > a_i$, i.e.,

$$r\Big((\frac{\alpha A_i}{r})^{\frac{1}{1-\alpha}} - S_i\Big) < f_i A_i (\frac{\alpha A_i}{r})^{\frac{\alpha}{1-\alpha}}$$
$$\Leftrightarrow \quad \Big(\frac{r}{\alpha A_i S_i^{\alpha-1}}\Big)^{\frac{1}{1-\alpha}} > 1 - \frac{f_i}{\alpha}.$$

So, the solution (k_i, a_i) is given by:

$$k_i = (\frac{\alpha A_i}{r})^{\frac{1}{1-\alpha}}$$
$$a_i = k_i - S_i.$$

In this case, agent i borrows (i.e $a_i > 0$) if and only if $\alpha A_i S_i^{\alpha-1} > r$ and lends if and only if $\alpha A_i S_i^{\alpha-1} \leq r$. $\square$

6.7 Proof of Proposition 6

Under our assumption $\bar{r}_1 > \bar{r}_2$, if $f_1 > \alpha$, then $f_2 < \alpha$.

We consider the following three cases.

Case 1: $f_1 < \alpha, \quad f_2 < \alpha$

Since $\bar{r}_1 > \bar{r}_2$, we have

$$\alpha A_1 S_1^{\alpha-1}\big(1 - \frac{f_1}{\alpha}\big)^{1-\alpha} > \alpha A_2 S_2^{\alpha-1}\big(1 - \frac{f_2}{\alpha}\big)^{1-\alpha}$$

$$\text{i.e., } \Big(\frac{A_2}{A_1}\Big)^{\frac{1}{1-\alpha}} < \frac{S_2}{S_1}\frac{1-\frac{f_1}{\alpha}}{1-\frac{f_2}{\alpha}}$$

Let (r, a_1, a_2, k_1, k_2) be an equilibrium. There exists an agent whose borrowing constraint is not binding. According to point 1 in Lemma 4, we have $r > min(\bar{r}_1, \bar{r}_2) = \bar{r}_2$. So, we will consider two cases: $\bar{r}_2 < r \leq \bar{r}_1$ and $r > \bar{r}_1$.

Case 1.1 $r > \bar{r}_1$. According to point 1 in Lemma 4, no borrowing constraint is binding. Hence, we find the same equilibrium as in the case without borrowing constraint:

$$k_i = \Big(\frac{\alpha A_i}{r}\Big)^{\frac{1}{1-\alpha}}, \quad a_i = k_i - S_i$$

$$r = \bar{r} \equiv \frac{\alpha\big(A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}\big)^{1-\alpha}}{(S_1+S_2)^{1-\alpha}}.$$

However, we have to check that both borrowing constraints are satisfied, i.e., $ra_i \leq f_i A_i k_i^{\alpha}$ for $i = 1, 2$. This condition is satisfied if and only if:

$$\frac{\alpha\big(A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}\big)^{1-\alpha}}{(S_1+S_2)^{1-\alpha}}\Big(\frac{A_i^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}(S_1+S_2) - S_i\Big) \leq f_i A_i\Big(\frac{A_i^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}(S_1+S_2)\Big)^{\alpha}$$

$$\Leftrightarrow (1 - \frac{f_i}{\alpha})\frac{A_i^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}} \leq \frac{S_i}{S_1+S_2}.$$

Condition $r > \bar{r}_i$ is equivalent to:

$$r = \frac{\alpha\big(A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}\big)^{1-\alpha}}{(S_1+S_2)^{1-\alpha}} > \alpha A_i S_i^{\alpha-1}\big(1 - \frac{f_i}{\alpha}\big)^{1-\alpha} = \bar{r}_1$$

$$\Leftrightarrow \frac{S_i}{S_1+S_2} > \frac{A_i^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}\big(1 - \frac{f_i}{\alpha}\big).$$

Notice that since $\bar{r}_1 > \bar{r}_2$, we have $\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\big(1-\frac{f_1}{\alpha}\big)} < \frac{S_2}{A_2^{\frac{1}{1-\alpha}}\big(1-\frac{f_2}{\alpha}\big)}$.

So, condition $\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\big(1-\frac{f_1}{\alpha}\big)} > \frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}}+A_2^{\frac{1}{1-\alpha}}}$ implies that $\frac{S_2}{A_2^{\frac{1}{1-\alpha}}\big(1-\frac{f_2}{\alpha}\big)} > \frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}}+A_2^{\frac{1}{1-\alpha}}}$.

We have

$$\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\big(1-\frac{f_1}{\alpha}\big)} > \frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}$$

Case 1.2 $\bar{r}_2 < r \leq \bar{r}_1$. According to Lemma 4, agent 1's borrowing constraint is binding and hence agent 2 is lender. It is easy to get that

$$k_1^{1-\alpha} - \frac{S_1}{k_1^\alpha} = \frac{f_1 A_1}{r}, \quad k_2 = \Big(\frac{\alpha A_2}{r}\Big)^{\frac{1}{1-\alpha}}$$
$$a_1 = k_1 - S_1, \quad a_2 = k_2 - S_2,$$

Financial market clearing condition:

$$a_1 + a_2 = 0 \tag{38}$$
$$\Leftrightarrow k_1 = k_1(r) \equiv S_1 + S_2 - \Big(\frac{\alpha A_2}{r}\Big)^{\frac{1}{1-\alpha}} \tag{39}$$

Since $k_1^{1-\alpha} - \frac{S_1}{k_1^\alpha} = \frac{f_1 A_1}{r}$, we have the following equation determining the equilibrium interest rate

$$\big(k_1(r)\big)^{1-\alpha} - \frac{S_1}{\big(k_1(r)\big)^\alpha} - \frac{f_1 A_1}{r} = 0 \tag{40}$$

$$\Leftrightarrow f(r) \equiv \Big(S_1 + S_2 - \Big(\frac{\alpha A_2}{r}\Big)^{\frac{1}{1-\alpha}}\Big)^{1-\alpha} - \frac{S_1}{\Big(S_1 + S_2 - \Big(\frac{\alpha A_2}{r}\Big)^{\frac{1}{1-\alpha}}\Big)^\alpha} - \frac{f_1 A_1}{r} = 0. \tag{41}$$

Since $k_1(r)$ is increasing in r, the function $f(r)$ is increasing in r.

We have $f(0) = -\infty$. Let r^* be defined by $S_1 + S_2 - \Big(\frac{\alpha A_2}{r^*}\Big)^{\frac{1}{1-\alpha}} = 0$. We have $f(r^*) = +\infty$. So, the equation $f(r) = 0$ has a unique solution and this solution is in $(0, r^*)$.

We now find condition under which

(i) $r \in (\bar{r}_2, \bar{r}_1]$.

(ii) $\mu_i \geq 0$, i.e., $\alpha A_1 k_1^{\alpha-1} \geq r$.

Notice that condition $\bar{r}_1 > \bar{r}$ implies that $k_1(\bar{r}_1) > 0$. So, we have $\bar{r}_2 < \bar{r}_1 < r^*$.

STEP 1. We firstly prove that $r > \bar{r}_2$. We have

$$k_1(\bar{r}_2) = S_1 + S_2 - \frac{S_2}{1 - \frac{f_2}{\alpha}}.$$

If $k_1(\bar{r}_2) \leq 0$ then $k_1(\bar{r}_2) \leq 0 < k_1(r)$. This implies that $\bar{r}_2 < r$.

If $k_1(\bar{r}_2) > 0$, it is easy to see that $f(k_1(\bar{r}_2)) < 0$, and hence $\bar{r}_2 < r$.

STEP 2. We will prove that $\alpha A_1 k_1^{\alpha-1} \geq r$ and $r \leq \bar{r}_1$.

We see that $\alpha A_1 k_1^{\alpha-1} \geq r$ is equivalent to

$$k_1 = S_1 + S_2 - \Big(\frac{\alpha A_2}{r}\Big)^{\frac{1}{1-\alpha}} < \Big(\frac{\alpha A_1}{r}\Big)^{\frac{1}{1-\alpha}} \Leftrightarrow r \leq \bar{r}.$$

Since $\bar{r}_1 \geq \bar{r}$, it is sufficient to prove that $r \leq \bar{r}$. We will do so by proving that $f(\bar{r}) \geq 0$. One can check that this is equivalent to:

$$\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\big(1 - \frac{f_1}{\alpha}\big)} \leq \frac{S_1 + S_2}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}} \tag{42}$$

We now need to verify that under this condition, there will be an equilibrium mentioned in the part 2 of Proposition 5. We can do so by verifying budget constraints, first-order and market clearing conditions.

It should be noticed that $r \in (\hat{r}_2, \bar{r}]$ in this case.

Case 2: $f_1 < \alpha, \quad f_2 > \alpha$

In this case, the borrowing constraint of agent 2 is non binding, and:

$$k_2 = \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}, \quad a_2 = k_2 - S_2$$

Case 2.1. $r \leq \bar{r}_1$.

This condition is satisfied if and only if $\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\left(1-\frac{f_1}{\alpha}\right)} > \frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}}+A_2^{\frac{1}{1-\alpha}}}$

In this case, the borrowing constraint of agent 1 is also not binding, we get the same equilibrium outcomes as in the unconstrained model.

Case 2.2. $r > \bar{r}_1$.

Using similar arguments as Case 1.2, we can find that the condition of parameters such that there exists an equilibrium in this case is:

$$\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\left(1-\frac{f_1}{\alpha}\right)} \leq \frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}}+A_2^{\frac{1}{1-\alpha}}}$$

And the equilibrium interest rate $r \in (\hat{r}_2, \bar{r}]$.

Combine Case 1 and Case 2 we see that:

If $\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\left(1-\frac{f_1}{\alpha}\right)} > \frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}}+A_2^{\frac{1}{1-\alpha}}}$, no borrowing constraints bind, and the equilibrium outcomes coincide with the unconstrained economy.

If $\frac{S_1}{A_1^{\frac{1}{1-\alpha}}\left(1-\frac{f_1}{\alpha}\right)} \leq \frac{S_1+S_2}{A_1^{\frac{1}{1-\alpha}}+A_2^{\frac{1}{1-\alpha}}}$, borrowing imposed on the borrower binds, and $r \in (\hat{r}_2, \bar{r}_1]$.

Case 3: $f_1 > \alpha, \quad f_2 > \alpha$, then no borrowing constraint is binding. We obtain the same general equilibrium outcomes as in the unconstrained economy.

Combine all three cases, we can derive results of Proposition 6:

If $\frac{A_1^{\frac{1}{1-\alpha}}+A_2^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}}}\frac{S_1}{S_1+S_2} + \frac{f_1}{\alpha} > 1$, no borrowing constraints bind, and the equilibrium outcomes coincide with the unconstrained economy.

If $\frac{A_1^{\frac{1}{1-\alpha}}+A_2^{\frac{1}{1-\alpha}}}{A_1^{\frac{1}{1-\alpha}}}\frac{S_1}{S_1+S_2} + \frac{f_1}{\alpha} \leq 1$, borrowing imposed on the borrower binds, and $r \in (\hat{r}_2, \bar{r}_1]$. □

6.8 Proof of Corollary 5

With Cobb-Douglas technology, when the borrowing constraint binds, the equilibrium interest rate is determined by:

$$S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}} = \frac{f_1 A_1}{r}\big[S_1 + S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\big]^{\alpha}$$

$$\Leftrightarrow f_1 A_1 = r\Big[S_1 + S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big]^{1-\alpha} - S_1 r\Big[S_1 + S_2 - \big(\frac{\alpha A_2}{r}\big)^{\frac{1}{1-\alpha}}\Big]^{-\alpha}$$

Denote $g(f_1, r) \equiv r\Big[S_1 + S_2 - (\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\Big]^{1-\alpha} - S_1 r\Big[S_1 + S_2 - (\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\Big]^{-\alpha} - f_1 A_1$

Then we have $g(f_1, r) = 0$, and the first derivative of $g(f_1, r)$ with respect to r and f_1 are given by:

$$g'(r) = \frac{\frac{\alpha S_1}{1-\alpha}(\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}} + S_2}{S_1 + S_2 - (\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}}\Big[S_1 + S_2 - (\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\Big]^{-\alpha} > 0$$

$$g'(f_1) = -A_1 < 0$$

By implicit theorem, $\frac{dr}{df_1} = -\frac{g'(f_1)}{g'(r)} > 0$, i.e, r is an increasing function in f_1. □

6.9 Proof of Corollary 6.

1. We have: $c_2 = A_2\Big(\frac{\alpha A_2}{r}\Big)^{\frac{\alpha}{1-\alpha}} + r\Big[S_2 - \Big(\frac{\alpha A_2}{r}\Big)^{\frac{1}{1-\alpha}}\Big]$. Then:

$$c_2'(r) = S_2\Big[1 - \Big(\frac{\alpha A_2 S_2^{\alpha-1}}{r}\Big)^{\frac{1}{1-\alpha}}\Big]$$

   Since $\hat{r}_2 = \alpha A_2 S_2^{\alpha-1} \leq r$ at the equilibrium when borrowing constraint of the borrower binds, we thus have $c_2'(r) \geq 0$. As r is an increasing function in f_1, we thus see that the the lender's consumption is an increasing function in f_1.

2. We have: $c_1 = A_1\big[S_1 + S_2 - (\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\big]^{\alpha} - r\big[S_2 - (\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\big]$. Thus, one can check that:

$$c_1'(r) = \frac{\alpha}{(1-\alpha)}(\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\Big[\frac{A_1}{r}\Big(S_1 + S_2 - (\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\Big)^{\alpha-1} - 1\Big] - S_2 \tag{43}$$

   Let $P(r) \equiv \frac{A_1}{r}\Big(S_1 + S_2 - (\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\Big)^{\alpha-1} - 1$

   It is easy to see that $P(r)$ is a decreasing function in r. From Proposition 5, we know that when the borrowing constraint is binding, $r \in [\hat{r}_2, \bar{r})$. Hence, $P(r) \geq P(\bar{r})$ for any r.

   We have: $P(\bar{r}) = \frac{1}{\alpha} - 1 > 0$

   Thus $P(r) \equiv \frac{A_1}{r}\Big(S_1 + S_2 - (\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\Big)^{\alpha-1} - 1 > 0$ for any $r \in [\hat{r}_2, \bar{r})$. This implies that $c_1'(r) = \frac{\alpha}{(1-\alpha)}(\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\Big[\frac{A_1}{r}\Big(S_1+S_2-(\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\Big)^{\alpha-1} - 1\Big] - S_2$ is a decreasing function in r when $r \in (\hat{r}_2, \bar{r})$

   Let us recall that at equilibrium when the borrowing constraint binds we have:

$$\frac{S_1}{A_1^{\frac{1}{1-\alpha}}(1 - \frac{f_1}{\alpha})} \leq \frac{S_1 + S_2}{A_1^{\frac{1}{1-\alpha}} + A_2^{\frac{1}{1-\alpha}}}$$

   One can derive from this condition that: $\frac{A_1}{A_2}\Big(\frac{S_2}{S_1}\Big)^{1-\alpha} > 1$

Therefore:

$$c_1'(\alpha A_2 S_2^{\alpha-1}) = \frac{S_2}{1-\alpha}\Big(\frac{A_1}{A_2}\big(\frac{S_2}{S_1}\big)^{1-\alpha} - 1\Big) > 0$$

As we already proved that $Y'(\bar{r}) = 0$, and $c_2'(\bar{r}) > 0$, we thus have:

$$c_1'(\bar{r}) = Y'(\bar{r}) - c_2(\bar{r}) < 0$$

Having proved that $c_1'(r)$ is a decreasing function in r for $r \in (\bar{r}_2, \bar{r})$, $c_1'(\hat{r}_2) > 0$, and $c_1'(\bar{r}) < 0$, we conclude that there exist a value $\tilde{r}$ such that $c_1'(r)$ is positive if $r \in (\bar{r}_2, \tilde{r})$ and negative if $r \in (\tilde{r}, \bar{r})$. As a result, $c_1(r)$ is increasing in r (or f_1) if $r \in (\bar{r}_2, \tilde{r})$ and is decreasing in r (or f_1) if $r \in (\tilde{r}, \bar{r})$.

We have: $Y = A_1\big[S_1 + S_2 - (\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\big]^{\alpha} + A_2(\frac{\alpha A_2}{r})^{\frac{\alpha}{1-\alpha}}$. Then:

$$Y'(r) = \frac{\alpha}{1-\alpha}(\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\big[A_1\Big(S_1 + S_2 - (\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\Big)^{\alpha-1} - A_2\big]$$

Therefore,

$$\begin{aligned} Y'(r) > 0 &\Leftrightarrow \Big(S_1 + S_2 - (\frac{\alpha A_2}{r})^{\frac{1}{1-\alpha}}\Big)^{\alpha-1}\frac{\alpha A_2}{r} - A_2 > 0 \\ &\Leftrightarrow r < \frac{\alpha\big((A_1)^{\frac{1}{1-\alpha}} + (A_2)^{\frac{1}{1-\alpha}}\big)^{1-\alpha}}{(S_1+S_2)^{1-\alpha}} \\ &\Leftrightarrow r < \bar{r} \end{aligned}$$

As we know from the Proposition 5, the interest rate in the case where borrowing constraints binds is always smaller than or equal to that in case without borrowing constraint. We thus conclude that $Y'(r)$ is always positive or $Y(r)$ is a increasing function of r, and as a result, is an increasing function of f_1.

□

6.10 Proof of Corollary 7.

1. The proof is straightforward. As already proved in Proposition 5, when the borrowing constraint imposed on the borrower binds, $r \in (\hat{r}_2, \bar{r}]$, where $\bar{r}$ is the interest rate in an economy without financial frictions.

2. As we already proved in Corollary 6, $c_2(r)$ is an increasing function in r. We also proved that $r \leq \bar{r}$ when the borrowing constraint on the borrower binds. Hence $c_2(r) \leq c_2(\bar{r})$, for any $r \in (\hat{r}_2, \bar{r}]$. Since $c_2(\bar{r})$ equals to the lender's consumption in the case without borrowing constrained, it follows that the lender's consumption in case with borrowing constraints is always smaller than or equal to that in case without borrowing constraints.

3. From Corollary 6, we know that when borrowing constraint imposed on the borrower binds, there exist $\tilde{r} \in (\hat{r}_2, \bar{r}]$ such that, $c_1(r)$ is decreasing in r for $r \in [\tilde{r}, \bar{r}]$, and r is sufficiently high. Since r is an increasing function of f_1, we thus conclude that the borrower's consumption in a constrained economy could be higher than that in a non-constrained economy if the collateral is binding but not very strict.

4. As we proved in Corollary 6, $Y(r)$ is an increasing function in r for $r \in (\hat{r}_2, \bar{r}]$. Therefore $Y(r) \leq Y(\bar{r})$ for any $r \in (\hat{r}_2, \bar{r}]$.

   One can easily check that $Y(\bar{r})$ equals to the equilibrium aggregate output when there are no borrowing constraints. Thus, the aggregate output in the case with borrowing constraints is always smaller than or equal to that in case without borrowing constraints.

□

6.11 Proof of Proposition 7.

Proof. As we already discussed in the beginning of section 3, at equilibrium $A_H \leq r \leq A_G$. Hence, there are possible cases as follows.

1. If $r = A_H < A_F < A_G$

   As $A_G > A_F > r$, according to Lemma 3, countries F and G borrows from the financial market and their borrowing constraints are binding, with:

$$a_i = \frac{f_i A_i}{r - f_i A_i} S_i, \quad k_i = \frac{r}{r - f_i A_i} S_i \quad \text{for} \quad i \in \{F, G\}$$

   As $A_H = r$, according to Lemma 3, countries H lends in the financial market, with:

$$-S_H \leq a_H \leq f_H(a_H + S_H), \quad k_H = a_H + S_H$$

   The market clearing condition requires:

$$\begin{aligned} a_H &= -(a_F + a_G) \\ \Leftrightarrow a_H &= -\frac{f_F A_F}{r - f_F A_F} S_F - \frac{f_G A_G}{r - f_G A_G} S_G \end{aligned}$$

   This implies that: $k_H = S_H - \frac{f_F A_F}{r - f_F A_F} S_F - \frac{f_G A_G}{r - f_G A_G} S_G$

   We need to check the condition $k_H \geq 0$. This condition is satisfied if and only if:

$$S_H \geq \frac{f_F A_F}{r - f_F A_F} S_F + \frac{f_G A_G}{r - f_G A_G} S_G \tag{44}$$

   Thus, we proved the last part of Proposition 7

2. If $A_H < r < A_F < A_G$

   As $A_G > A_F > r$, according to Lemma 3, countries F and G borrows from the financial market and their borrowing constraints are binding, with:

$$a_i = \frac{f_i A_i}{r - f_i A_i} S_i, \quad k_i = \frac{r}{r - f_i A_i} S_i \quad \text{for} \quad i \in \{F, G\}$$

   As $A_H < r$, according to Lemma 3, country H lends all his saving to the financial market. We have:, country G and country F borrows from the international market.

$$a_H = -S_H, \quad k_H = 0$$

The world interest rate is determined by:

$$S_H = \frac{f_F A_F}{r - f_F A_F} S_F + \frac{f_G A_G}{r - f_G A_G} S_G \tag{45}$$

We see that the left-hand side of (45) is a decreasing function in r. Hence, this equation has a solution r such that $A_H < r < A_F < A_G$ if and only if:

$$\frac{f_F A_F}{A_F - f_F A_F} S_F + \frac{f_G A_G}{A_F - f_G A_G} S_G < S_H < \frac{f_F A_F}{A_H - f_F A_F} S_F + \frac{f_G A_G}{A_H - f_G A_G} S_G$$

Thus, we proved the 5th part of Proposition 7

3. If $r = A_F < A_G$

   As $A_G > r$, according to Lemma 3, country G borrows from the financial market and its borrowing constraint is binding, with:

$$a_G = \frac{f_G A_G}{A_F - f_G A_i} S_G, \quad k_G = \frac{A_F}{A_F - f_G A_G} S_G$$

   As $A_H < r$, according to Lemma 3, country H lends all his saving to the financial market. We have:, country G and country F borrows from the international market.

$$a_H = -S_H, \quad k_H = 0$$

   As $A_F = r$, according to Lemma 3, the solution to the country F's maximization problem is any (a_F, k_F) such that: $-S_F \leq a_F \leq f_F(a_F + S_F), k_F = a_F + S_F$

   The market clearing conditions requires that at equilibrium:

$$a_F = -(a_H + a_G)$$
$$\Leftrightarrow a_F = S_H - \frac{f_G A_G}{r - f_G A_G} S_G$$

   This implies that: $k_F = S_F + S_H - \dfrac{f_G A_G}{A_F - f_G A_G} S_G$

   - Country F borrows in the financial market if and only if the following conditions are satisfied:

$$\begin{cases} a_F > 0 \\ r a_F \leq f_F A_F k_F \end{cases} \Leftrightarrow \begin{cases} S_H > \dfrac{f_G A_G}{A_F - f_G A_G} S_G \\ S_H \leq \dfrac{f_G A_G}{A_F - f_G A_G} S_G + \dfrac{f_F}{1 - f_F} S_F \end{cases}$$

     In short, country F borrows in the financial market if:

$$\frac{f_G A_G}{A_F - f_G A_G} S_G < S_H \leq \frac{f_G A_G}{A_F - f_G A_G} S_G + \frac{f_F}{1 - f_F} S_F$$

     Thus, we proved the 4th part of Proposition 7

- Country F lends in the financial market if and only if the following conditions are satisfied:

$$\begin{cases} a_F \leq 0 \\ k_F > 0 \end{cases} \Leftrightarrow \begin{cases} S_H \leq \dfrac{f_G A_G}{A_F - f_G A_G} S_G \\ S_H > \dfrac{f_G A_G}{A_F - f_G A_G} S_G - S_F \end{cases}$$

In short, country F lends in the financial market if:

$$\frac{f_G A_G}{A_F - f_G A_G} S_G - S_F < S_H \leq \frac{f_G A_G}{A_F - f_G A_G} S_G$$

Thus, we proved the 3rd part of Proposition 7.

4. $A_F < r < A_G$

As $A_H < A_F < r$, according to Lemma 3, both of these countries lend all their savings to the financial market. We have:

$$a_i = -S_i, \quad k_i = 0 \quad \text{for} \quad i \in \{H, F\} \tag{46}$$

As $A_G > r$, according to Lemma 3, country G borrows from the financial market, and its borrowing constraint is binding. we have:

$$k_G = \frac{r}{r - f_G A_G} S_G, \quad a_G = \frac{f_G A_G}{r - f_G A_G} S_G$$

The equilibrium interest rate is given by:

$$\frac{f_G A_G}{r - f_G A_G} S_G = S_H + S_F \Leftrightarrow r = f_G A_G \Big(1 + \frac{S_G}{S_H + S_F} \Big)$$

Hence, $r \in (A_F, A_G)$ if and only if:

$$\frac{f_G}{1 - f_G} S_G - S_F < S_H < \frac{f_G A_G}{A_F - f_G A_G} S_G - S_F$$

Thus, we proved the 2nd part of Proposition 7.

5. $A_F < r = A_G$

As $A_H < A_F < r$, according to Lemma 3, both of these countries lend all their savings to the financial market. We have:

$$a_i = -S_i, k_i = 0 \quad \text{for} \quad i \in \{H, F\} \tag{47}$$

As $A_G = r$, according to Lemma 3, the solution to the country G's maximization problem is:

The market clearing condition requires that:

$$\begin{aligned} a_G &= -(a_H + a_F) \\ \Leftrightarrow a_G &= S_H + S_F \end{aligned}$$

This implies that: $k_G = \sum_i S_i \quad for \quad i \in \{H, F, G\}$

We need to check the condition $ra_G \leq f_G A_G k_G$.One can verify that this condition is satisfied when

$$S_H \leq \frac{f_G}{1-f_G} S_G - S_F \tag{48}$$

Thus, we proved the 1st part of Proposition 7.

□

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