

ON THE MAXIMIN PRINCIPLE & THE RATE OF DISCOUNT: A SIMPLE DYNAMIC PROGRAMMING ARGUMENT*

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ABSTRACT

The article establishes a dynamic programming argument for a maximin problem in the line of Gilboa-Schmeidler, where priors are defined on compact set of discount rates. Even though the consideration of a maximin criterion results in a program that is not convex and not stationary over time, it is proved that a careful reference to extended dynamic programming principles and a minmax functional equation however allows for circumventing these difficulties and recovering an optimal sequence that is time consistent. This in its turn brings about a stationary dynamic programming argument.

Keywords: maximin principle, non-convexities, value function, policy function, supermodularity.

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1. INTRODUCTION

Dating from Stokey & Lucas [13], dynamic programming techniques and the Bellman principle of optimality have been developed in order to facilitate the analysis of inter-temporal equilibria. The traditional approach to dynamic programming in economics has been set forth by Stokey & Lucas [13]. It is based upon the well known Ramsey separable criterion that takes a constant discount factor $\delta \in]0, 1[$ as a given. For every indirect utility V , a production correspondence Γ and a given x_0 , the value of the payoff is given by

$$\sup_{x_{t+1} \in \Gamma(x_t)} \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}).$$

Letting $J(x_0)$ denote the value of this program, the dynamic programming approach indeed establishes that its solution $\{x_1^*, x_2^*, \dots, x_t^*, \dots\}$ is to obey $J(x_0) = V(x_0, x_1^*) + \delta J(x_1^*)$ and $J(x_t^*) = V(x_t^*, x_{t+1}^*) + \delta J(x_{t+1}^*)$ for every $t \geq 1$, *i.e.*, allow to study complex infinite horizon problems through sequences of stationary two-period problems.

Keeping this approach in mind, the current article studies the implications of a change of focus to a maximin problem. Early examples in the economics literature are the multiple priors formulation of Gilboa and Schmeidler [7]:

$$\sup_{\pi \in \Delta} \inf \sum_{t=0}^{\infty} \pi_t V(x_t, x_{t+1}), \tag{GS}$$

where Δ depicts the set of probability measures, $\pi_t \in]0, 1[$, for $\sum_{t=0}^{\infty} \pi_t < \infty$, and where the sequence $\{\pi_t\}_{t=0}^{+\infty}$ is stationary. The operator *min* of a formulation referring to a *worst-scenario* case and the maximin operator having been shown by

Gilboa and Schmeidler [7] to be consistent with *global ambiguity aversion*, the authors of the current article are more specifically interested in building a multiple priors approach that wouldn't, as for these authors, revolve around around the set of the states of the world in an uncertain environment but rather, and in a deterministic environment, upon the set of dates, or time, and a sequence of discount rates.

Given a compact set of discount factors $\mathcal{D} = [\underline{\delta}, \bar{\delta}]$, a first hint would build upon a simple reformulated fixed discount Ramsey-like problem: $\sup_{x_{t+1} \in \Gamma(x_t)} \inf_{\delta \in \mathcal{D}} \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1})$. This criterion being however monotonic in δ , the solution of this minimization problem is trivially given by $\underline{\delta}$ and this reformulation remains unsatisfactory in solely building upon a redefinition of the given discount factor. In order to circumvent these insufficiencies one must hence rely upon a discount function which is not monotonic.

This article shall consider a family of time discount factors* like $\delta_t = (1 - \delta)\delta^t$, where δ belongs to an interval $[\underline{\delta}, \bar{\delta}]$. The term $(1 - \delta)$ is chosen to give an arbitrage, and to normalize the sum of discount factors to 1. This leads to the following criterion

$$\sup_{x_{t+1} \in \Gamma(x_t)} \inf_{\delta \in \mathcal{D}} (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}). \quad (\text{MM})$$

The aim of this article is then to provide a benchmark argument that characterizes the solutions of program (MM) through a *stationary* dynamic programming approach. Interestingly, even though the preferences (MM) characterized by a maximin criterion are not stationary and the infimum part in δ of problem (MM) is not convex—the optimal discount factor δ^* may indeed differ between two periods—this article establishes a set of formal conditions under which the optimal decision rule is stationary and the optimal sequence satisfies a stationary "Bellman-like" equation.

This functional equation is further proved to assume a unique solution. In order to prove that this solution corresponds to the value function of the program

*A non monotonic convenient specification could have been $\sup_{x_{t+1} \in \Gamma(x_t)} \inf_{\{\delta_t\}_{t=0}^{\infty} \in \mathcal{D}^{\infty}} \sum_{t=0}^{\infty} (1 - \delta_t)\delta_t V(x_t, x_{t+1})$. It is more interesting, but the weights are non stationary and do not correspond to the Gilboa & Schmeidler's [7] (GS)-type problem. Note further that if $\delta_t = \delta$, the sum $\sum_{t=0}^{\infty} (1 - \delta)\delta$ tends to infinity and does not fit either with the Gilboa & Schmeidler [7] framework.

(MM), the arguments proceed in four steps. (i) The existence of a unique optimal policy function is first established. (ii) Under an extra super modularity assumption on the payoff V , one establishes the monotonicity of the optimal sequence. (iii) A decisive step proves that for any $x_0 > 0$, there exists a unique $\delta^* \in \mathcal{D}$ such that the optimal $\chi^* = \{x_t^*\}_{t=0}^\infty$ sequence is the solution of a classical fixed discount Ramsey problem with a discount factor δ^* . Once the problem of the optimal δ^* is solved, the optimal capital sequence $\chi^*(\delta^*)$ is considered and proved to satisfy the Bellman-like equation. (iv) The fixed point of that functional equation is eventually proved to be the value function of maximin problem. To finish, the article proves that the problems maximin and minimax are equivalent.

This article hence provides a generalization of the classical results of Stokey-Lucas [13], but also of more recent results in dynamic programming theory that can cope with non-convex structures such as the one recently put forth by Kahimigashi [14]. A key advance of this article is an optimally determined sequence of discount rates is compatible with stationary dynamic programming tools.

One may further notice that the dynamic programming argument of this article may be used even in the case of a non-connected set of discount rate for which a time-consistent decision rule would keep on being available.

The article is organized as follows. Section 2 introduces the maximin problem. Section 3 completes a functional approach and a stationary dynamic programming argument for the bounded case. This is extended in Section 4 to payoffs unbounded from below. Section 5 illustrates how a continuous time argument would exhibit drastically different properties. Section 6 compares with various branches of the earlier literature. Some parametric examples are considered in Section 7. Proofs are gathered in the Appendix.

2. A MAXIMIN PROBLEM

Consider an economy endowed with a technology defined by a correspondence $\Gamma : \mathbb{R}_+ \rightarrow \mathcal{P}(\mathbb{R}_+)$. For each $x_0 \geq 0$, define $\Pi(x_0)$ as the set of feasible sequences $\chi = \{x_t\}_{t=0}^\infty$ satisfying $x_{t+1} \in \Gamma(x_t)$ for any $t \geq 0$. Further suppose that the set of

eligible discount factors in this economy is $\mathcal{D} = [\underline{\delta}, \bar{\delta}]$, with $0 < \underline{\delta} < \bar{\delta} < 1$.

Given $x_0 > 0$, consider the maximin problem

$$\sup_{\chi \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} \left[(1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) \right], \quad (\text{MM})$$

where $V : \text{Graph}(\Gamma) \rightarrow \mathbb{R} \cup \{-\infty\}$. One may first wonder whether it is possible to ease and reverse the treatment of the optimisation program (MM) and first consider its supremum component with respect to χ *before* infimum component with respect to x . Otherwise stated, is it possible to establish that

$$\sup_{\chi \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} v(\delta, \chi) = \inf_{\delta \in \mathcal{D}} \sup_{\chi \in \Pi(x_0)} v(\delta, \chi).$$

Though the answer is not trivial, the following lemma can be established:

LEMMA 1. [SUPINF VS INFSUP] *For given x_0, Γ, V and \mathcal{D} , the supinf and the infsup problems relate as follows:*

$$\sup_{\chi \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} v(\delta, \chi) \leq \inf_{\delta \in \mathcal{D}} \sup_{\chi \in \Pi(x_0)} v(\delta, \chi).$$

REMARK 1. *For every (δ, χ) , since $v(\delta, \chi) = (1 - \delta)V(x_0, x_1) + \delta v(\delta, \chi_1)$, one may notice that:*

$$\begin{aligned} \frac{\partial v(\delta, \chi)}{\partial \delta} &= -V(x_0, x_1) + v(\delta, \chi_1) + \delta \frac{\partial v(\delta, \chi_1)}{\partial \delta} \\ &= -V(x_0, x_1) + v(\delta, \chi_1) + \delta V(x_1, x_2) + \delta v(\delta, \chi_2) + \delta \frac{\partial v(\delta, \chi_2)}{\partial \delta} \\ &= \dots \\ &= - \sum_{t=0}^T \delta^t V(x_t, x_{t+1}) + \sum_{t=0}^T \delta^t v(\delta, \chi_{t+1}) + \delta^T \frac{\partial v(\delta, \chi_{T+1})}{\partial \delta}. \end{aligned}$$

Under suitable conditions, e.g., when V is bounded from below, $\lim_{T \rightarrow \infty} \delta^T (\partial v(\delta, \chi_{T+1}) / \partial \delta) = 0$, hence obtaining

$$\frac{\partial v(\delta, \chi)}{\partial \delta} = - \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) + \sum_{t=0}^{\infty} \delta^t v(\delta, \chi_{t+1}).$$

Due to the lack of convexity with respect to δ of the infimum component of the program (MM), the uniqueness or the determination of the optimal discount rate are however out of reach. It is worthwhile noticing that, would the convexity with respect to δ be recovered, the solution of the infimum problem would correspond to the selection of an optimal singleton δ^ and the supinf and the infsup problems would be equivalent.*

Define $v(\delta, \chi)$ as the operand in the program (MM):

$$v(\delta, \chi) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}).$$

For every feasible sequence $\chi \in \Pi(x_0)$, further let the function $\hat{v}(\cdot)$ feature its infimum with respect to the discount factor:

$$\hat{v}(\chi) = \inf_{\delta \in \mathcal{D}} v(\delta, \chi).$$

Further denote by $\text{dom}(\hat{v}, x_0)$ the set of feasible sequences $\chi \in \Pi(x_0)$ such that $\hat{v}(\chi) > -\infty$ for a given $x_0 \geq 0$. A range of classical assumptions is first to be introduced on the problem (MM).

ASSUMPTION T1. Γ is non-empty, compact-valued and continuous on \mathbb{R}_+ .

ASSUMPTION T2. The graph of Γ is convex, i.e for any x, x' and $y \in \Gamma(x), y' \in \Gamma(x')$, for any $0 \leq \lambda \leq 1$, $(1 - \lambda)y + \lambda y' \in \Gamma((1 - \lambda)x + \lambda x')$.

ASSUMPTION V1. The payoff function V is continuous on $\text{Gr}(\Gamma)$.

ASSUMPTION V2. The payoff function V is strictly concave on $\text{int}[\text{Gr}(\Gamma)]$.

The existence and the uniqueness of the problem (MM), or equivalently of the maximization of $\hat{v}(\chi_t)$ for $\chi_t \in \Pi(x_0)$, are then faced with in the following statement:

PROPOSITION 1. Consider the functional \hat{v} under Assumptions T1 and V1.

- (i) An optimal χ^* does exist.
- (ii) Further add Assumptions T2 and V2,
 - a/ \hat{v} is strictly concave on $\text{dom}(\hat{v}, x_0)$.
 - b/ If $\sup_{\chi \in \Pi(x_0)} \hat{v}(\chi) > -\infty$, the maximin problem (MM) has a unique solution $\chi^* \in \Pi(x_0)$ such that

$$\begin{aligned} \hat{v}(\chi^*) &= \sup_{\chi \in \Pi(x_0)} \hat{v}(\chi) \\ &= \sup_{\chi \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) \\ &= \max_{\chi \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}). \end{aligned}$$

3. THE FUNCTIONAL EQUATION APPROACH & THE DYNAMIC PROGRAMMING ARGUMENT

The value function of the program (MM) is defined by

$$J(x_0) = \sup_{\chi \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} \left[(1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}) \right].$$

Then observe that, in line with the approach of section 2, the definition of the value function reformulates to:

$$J(x_0) = \sup_{\chi \in \Pi(x_0)} \hat{v}(\chi). \tag{1}$$

One may first notice that the retainment of the following assumption on the payoff will suffice for ensuring that the solution of the problem (MM) is bounded from below and $J(x_0) > -\infty$ for any x_0 in Proposition 1:

ASSUMPTION V3. The payoff function V is bounded from below.

For a feasible sequence $\chi \in \Pi(x_0)$, further define $\chi_t = (x_t, x_{t+1}, x_{t+2}, \dots)$ as its truncated formulation that starts at $t \geq 1$. Observe that the operand $v(\delta, \chi)$ of the optimisation program (MM) can be split into a component that describes the current payoff and another one that features this discounted operand at a later date:

$$\begin{aligned} v(\delta, \chi_0) &= (1 - \delta)V(x_0, x_1) + \delta v(\delta, \chi_1) \\ &= (1 - \delta)V(x_0, x_1) + (1 - \delta)\delta V(x_1, x_2) + \delta^2 v(\delta, \chi_2) \\ &= \dots \\ &= (1 - \delta) \sum_{t=0}^T \delta^t V(x_t, x_{t+1}) + \delta^{T+1} v(\delta, \chi_{T+1}). \end{aligned}$$

Likewise and from the definition of $\hat{v}(\cdot)$,

$$\hat{v}(\chi_0) = \inf_{\delta \in \mathcal{D}} \left\{ (1 - \delta)V(x_0, x_1) + \delta v(\chi_1, \delta) \right\}.$$

Note however that the obtention of a Bellman-like formulation with an optimal δ like

$$\hat{v}(\chi_0) = \left\{ (1 - \delta^*)V(x_0, x_1) + \delta^* \hat{v}(\chi_1) \right\}$$

is everything but trivial as this would rest upon the hypothetical existence of a $\delta^* \in \mathcal{D}$ such that:

$$\delta^* \in \underset{\delta \in \mathcal{D}}{\operatorname{arg\,inf}} v(\chi_0, \delta) \cap \underset{\delta \in \mathcal{D}}{\operatorname{arg\,inf}} v(\chi_1, \delta).$$

This can however not be guaranteed due to the fail on convexity of the problem (MM).

As this closely mimics the features of the classical dynamic programming argument considered in Stokey & Lucas [13], one may however wonder if such a line of reasoning transposes to the supremum of this functional \hat{v} on $\operatorname{dom}(\hat{v}, x_0)$ as it is defined by the value function J . Does it assume such a representation and can it be recovered as the solution of a functional equation? The following result provides a first range of results in establishing that a function $W(\cdot)$ can indeed be recovered as the solution of related minmax functional equation or, equivalently, as the fixed point of a minmax operator:

LEMMA 2. [A MINMAX BELLMAN-LIKE OPERATOR] *Under Assumptions , T1-T2 and V1-V3:*

- (i) *There exists a unique continuous function $W : \mathbb{R}_+ \rightarrow \mathbb{R}$ which is a solution of the following Bellman-like functional equation:*

$$W(x_0) = \sup_{x_1 \in \Gamma(x_0)} \inf_{\delta \in \mathcal{D}} \left\{ (1 - \delta)V(x_0, x_1) + \delta W(x_1) \right\}.$$

- (ii) *The function W is strictly increasing and concave.*

One may now wonder whether this solution W is equivalent to the value function J and whether one can recover the *Principle of Optimality*. Otherwisel stated, for any V, Γ, \mathcal{D} and W , does

$$W(x_0) = \sup_{y \in \Gamma(x_0)} \inf_{\delta \in \mathcal{D}} \left\{ (1 - \delta)V(x_0, y) + \delta W(y) \right\}$$

imply that $W(x_0) = \sup_{\chi \in \Pi(x_0)} \hat{v}(\chi)$, aka the very definition of the value function $J(x_0)$?

The following proposition is important in that quest since it first proves that under, some extra assumptions, the W -optimal policy function φ is monotonic. This monotonicity property will indeed later play a major role in the establishment of

the stationarity and time-consistency of the decision rule and the equality between the value function J and the solution of the functional equation W .

It is based upon the adoption of the following set of fairly classical assumptions:

ASSUMPTION T₃. For any $0 \leq x \leq x'$, $\Gamma(x) \subset \Gamma(x')$.

ASSUMPTION T₄. For x sufficiently small, $x \in \text{int}(\Gamma(x))$.

ASSUMPTION T₅. There exist $a, b > 0$ such that $0 < a < 1$ and for any $x \geq 0$, $y \in \Gamma(x)$, $y \leq ax + b$.

ASSUMPTION V₄. The payoff function V is strictly increasing in its first variable and decreasing in its second one.

Another more specific qualification on the payoff function will facilitate the establishment of the monotonicity of the policy function.

ASSUMPTION V₅. The function V is super modular.[†]

PROPOSITION 2. [THE POLICY FUNCTION] *Consider the previous environment under Assumptions T₁-T₅ and V₁-V₅.*

(i) *For every $x_0 > 0$, there exists a unique $x_1^* \in \Gamma(x_0)$ such that*

$$W(x_0) = \min_{\delta \in \mathcal{D}} \left\{ (1 - \delta)V(x_0, x_1^*) + \delta W(x_1^*) \right\}.$$

(ii) *Let the W -policy function φ be defined by:*

$$\begin{aligned} \varphi(x_0) &= \operatorname{argmax}_{z \in \Gamma(x_0)} w(x_0, z); \\ w(x_0, z) &= \min_{\delta \in \mathcal{D}} \left\{ (1 - \delta)V(x_0, z) + \delta W(z) \right\}, \end{aligned}$$

it is strictly increasing.

(iii) *Define $x_t^* = \varphi^t(x_0)$ as the optimal sequence, or $x_{t+1}^* = \varphi(x_t^*)$ for any $t \geq 0$. The optimal sequence $\chi^* = \{x_t^*\}_{t=0}^\infty$ is monotonic.*

[†]*I.e., for every (x, x') and (y, y') that belong to $\text{Graph}(\Gamma)$, $V(x, y) + V(x', y') \geq V(x', y) + V(x, y')$ prevails whenever $(x', y') \geq (x, y)$. If V is twice differentiable, super modularity is equivalent to the property that cross derivatives are always positive.*

The key Proposition 3 will now prove how, for every $x_0 > 0$, there exists $\delta^* \in \mathcal{D}$, with $\chi^* = \{x_0, \varphi(x_0), \varphi^2(x_0), \dots, \varphi^t(x_0), \dots\}$, such that optimal sequence χ^* is a solution to an optimization program with a fixed discount factor δ^* . For that purpose and for every $\delta \in \mathcal{D}$, let $\mathcal{R}(\delta)$ denote such a benchmark program, henceforward referred to as the corresponding fixed discount *Ramsey problem*:

$$\sup_{\chi \in \Pi(x_0)} (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}). \quad (\mathcal{R}(\delta))$$

PROPOSITION 3. [AN EXTENDED PRINCIPLE OF OPTIMALITY] *Under assumptions V1-V4, T1-T5, consider, for a given x_0 , the optimal sequence $\chi^* = \{x_t^*\}_{t=0}^{\infty} = \{\varphi^t(x_0)\}_{t=0}^{\infty}$ as it is defined from Proposition 2(iii).*

- (i) *There exists $\delta^* \in \mathcal{D}$ such that χ^* is solution of the fixed discount Ramsey problem $\mathcal{R}(\delta^*)$.*
- (ii) *More precisely,*
 - a) *If the sequence $\{x_t^*\}_{t=0}^{\infty}$ is strictly increasing, then it is a solution to $\mathcal{R}(\underline{\delta})$;*
 - b) *If the sequence $\{x_t^*\}_{t=0}^{\infty}$ is strictly decreasing, it is a solution to $\mathcal{R}(\bar{\delta})$;*
 - c) *If the sequence $\{x_t^*\}_{t=0}^{\infty}$ is constant, the corresponding value δ^* lies between $\underline{\delta}$ and $\bar{\delta}$.*

Theorem 1 is then a direct consequence of Propositions 2 and 3 and establishes the stationarity of the policy function and an augmented form of the *Principle of Optimality*.

THEOREM 1. [STATIONARY DECISION RULES & STATIONARY DYNAMIC PROGRAMMING] *Under assumptions T1-T5 and V1-V5, for any $x_0 > 0$, the following propositions are true:*

- (i) *The decision rule*

$$\varphi(x_0) = \operatorname{argmax}_{z \in \Gamma(x_0)} \min_{\delta \in \mathcal{D}} \{(1 - \delta)V(x_0, z) + \delta W(z)\}$$

is not time dependent, or stationary.

- (ii) *The value of the Maximin problem is equal to the value of the Minimax problem.*

$$\sup_{\chi \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} v(\delta, \chi) = \max_{\chi \in \Pi(x_0)} \min_{\delta \in \mathcal{D}} v(\delta, \chi)$$

$$\begin{aligned}
&= \inf_{\delta \in \mathcal{D}} \sup_{\chi \in \Pi(x_0)} v(\delta, \chi) \\
&= \min_{\delta \in \mathcal{D}} \max_{\chi \in \Pi(x_0)} v(\delta, \chi).
\end{aligned}$$

(iii) $J(x_0) = W(x_0)$.

Further remark that, under assumptions **T1-T5** and **V1-V5**, there exists a unique *golden rule* x^G which maximizes $V(x, x)$.

ASSUMPTION V6. The function $\psi(x) = -V_2(x, x)/V_1(x, x)$ is strictly increasing in $]0, x^G[$.

Under assumption **V6**, for each δ , the fixed discount Ramsey problem $\mathcal{R}(\delta)$ assumes a unique non trivial steady state x^δ , which is increasing with respect to δ , namely:

$$x^{\underline{\delta}} \leq x^\delta \leq x^{\bar{\delta}} \quad \text{for every } \delta \in \mathcal{D}.$$

If $x^{\underline{\delta}} \leq x_0 \leq x^{\bar{\delta}}$, the discount factor if solution to $V_2(x_0, x_0) + \delta V_1(x_0, x_0) = 0$. If $x_0 > x^{\bar{\delta}}$, the discount factor is $\delta^* = \bar{\delta}$. Once the problem of the optimal δ^* is solved, the optimal capital sequence $\chi^*(\delta^*)$ is considered and proved to satisfy the Bellman-like equation. The fixed point of that functional equation is equal to the value function.

COROLLARY 1. [STATIONARY DECISION RULES & LONG-RUN CONVERGENCE]
Under assumptions **T1-T5** and **V1-V6**,

- (i) For $x_0 < x^{\underline{\delta}}$, the optimal sequence is strictly increasing and converges to $x^{\underline{\delta}}$.
- (ii) For $x_0 > x^{\bar{\delta}}$, the optimal sequence is strictly decreasing and converges to $x^{\bar{\delta}}$.
- (iii) For $x^{\underline{\delta}} \leq x_0 \leq x^{\bar{\delta}}$, the optimal sequence is constant.

REMARK 2. It is to be mentioned that the whole argument can be extended to more general spaces. Even though, from the beginning and by the sake of simplicity, the analysis has been anchored on the set of positive real numbers, the same results are available for more general spaces. Consider indeed a reformulation of the **T** assumptions according to:

ASSUMPTION S1. Γ is non-empty, compact-value and continuous on X , where X a convex subset of positive orthant of a Banach space, equipped with a lattice structure à la Topkis [15] with a pre-order \leq satisfying for every $x \leq x'$, the interval $[x, x']$ is compact.

ASSUMPTION S2. Γ is ascending, i.e for any $x \leq x'$, $\Gamma(x) \subset \Gamma(x')$.

Instead of Assumptions T1 and T3, consider assumptions S1 and S2, and let the increasing property in V4 be understood in the meaning of pre-order ; the results as Propositions 1 and 2 as well as the ones Lemmas 1 and 2 keep on prevailing, the proofs being indeed exactly the same. One just has to pay attention at the proofs of claims (i) and (ii) of Proposition 2: instead of the interval $]y_0 - \epsilon, y_0 + \epsilon[$, make use of $B(y_0, \epsilon) \cap [x_0, z_0]$, where $B(y_0, \epsilon)$ is an open ball of radius ϵ , with $\epsilon > 0$ small enough. For each time one has to compare a vector x with a vector y : instead of \leq or $<$, make use \leq or \prec .

ASSUMPTION S3. X is totally ordered by the pre-order \leq .

With assumption S3, the result in Proposition 3 as well as Theorem 1 are available, the proofs being unchanged with respect to the current argument. The only difference formulates as the need of $x \in B(x_0, \epsilon)$ instead of $x_0 - \epsilon < x < x_0 + \epsilon$.

4. PAYOFFS UNBOUNDED FROM BELOW

In this section, Assumption V3 is relaxed and the payoff becomes unbounded from below. Because of this unboundedness, contracting map techniques cannot anymore be used to prove the existence of a solution to the functional Bellman-like equation. However, and thanks to the supplementary condition V6, the existence argument can be established by *guess and verify* methods.[‡]

Under the assumptions T1-T5 and V1-V2, V4-V6, for every δ there exists a unique long run steady state x^δ for the fixed discount Ramsey problem $\mathcal{R}(\delta)$, and $x^\delta \leq x^{\bar{\delta}} \leq x^{\delta}$, for every $\delta \in \mathcal{D}$. Define then $J_\delta(x_0)$ as the value function of $\mathcal{R}(\delta)$:

$$J_\delta(x_0) = \sup_{\chi \in \Pi(x_0)} (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t, x_{t+1}). \quad (2)$$

[‡]Observe however that it would be more difficult to apply these methods were one to relax Assumption V6, for there would then exist multiple steady solutions for $\mathcal{R}(\delta)$.

Consider now the function $W(\cdot)$ as defined by:

- (i) For $0 \leq x_0 \leq x^{\underline{\delta}}$, $W(x_0) = J_{\underline{\delta}}(x_0)$,
- (ii) For $x^{\underline{\delta}} \leq x_0 \leq x^{\bar{\delta}}$, take δ satisfying $V_2(x_0, x_0) + \delta V_1(x_0, x_0) = 0$, $W(x_0) = J_{\delta}(x_0) = V(x_0, x_0)$.
- (iii) For $x \geq x^{\bar{\delta}}$, $W(x_0) = J_{\bar{\delta}}(x_0)$.

Making use of the monotonicity of x^{δ} with $\delta \in \mathcal{D}$, the following preparation lemma can first be established,

LEMMA 3. Assume **T1-T5** and **V1-V2, V4-V6**. The function W is strictly concave.

THEOREM 2. Assume **T1-T5** and **V1-V2, V4-V6**.

- (i) The value of the Maximin problem is equal to the value of the Minimax problem

$$\begin{aligned} \sup_{\chi \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} v(\delta, \chi) &= \max_{\chi \in \Pi(x_0)} \min_{\delta \in \mathcal{D}} v(\delta, \chi) \\ &= \inf_{\delta \in \mathcal{D}} \sup_{\chi \in \Pi(x_0)} v(\delta, \chi) \\ &= \min_{\delta \in \mathcal{D}} \max_{\chi \in \Pi(x_0)} v(\delta, \chi). \end{aligned}$$

- (ii) $J(x_0) = W(x_0)$.

PROPOSITION 4. Assume **T1-T5** and **V1-V2, V4-V6**. The function W is solution of the functional equation

$$W(x_0) = \max_{x_1 \in \Gamma(x_0)} \min_{\delta \in \mathcal{D}} \{(1 - \delta)V(x_0, x_1) + \delta W(x_1)\}.$$

This characterization of payoffs unbounded from below will prove useful for the characterization of some benchmark examples in Section 7.

5. A CONTINUOUS TIME ARGUMENT

For the continuous time case, the operand is assumed to be given by

$$\int_0^{+\infty} \rho e^{-\rho t} U[x(t), \dot{x}(t)] dt.$$

Along the discrete time case, assume that the discount rate ρ can be changed in $\mathcal{D} = [\underline{\rho}, \bar{\rho}] \in]0, +\infty[$ while $(x(t), \dot{x}(t)) \in \Omega$, for $\Omega = \text{Gr}(\Gamma)$ some convex set with non empty interior. It is first noticed that, on average, this criterion is undiscounted:

$$\int_0^{+\infty} \rho e^{-\rho t} dt = 1, \quad \forall \rho > 0$$

Consider then a corresponding maximin problem:

$$\begin{aligned} \sup_{x(t)} \inf_{\rho \in \mathcal{D}} \int_0^{+\infty} \rho e^{-\rho t} U[x(t), \dot{x}(t)] dt. & \quad (\text{CTMM}) \\ \text{s.t. } (x(t), \dot{x}(t)) \in \Omega & \\ x(0) = x_0 \text{ given} & \end{aligned}$$

For every $\rho \in [\underline{\rho}, \bar{\rho}]$, for future reference and along the terminology of the discrete time argument, denote by $\mathcal{R}(\rho)$ the associated fixed discount supremum *Ramsey problem*.

Consider then the function defined over the admissible paths $\chi = (x(t))_{t \in \mathbb{R}}$:

$$v(\rho, \chi, \dot{\chi}) = \int_0^{+\infty} \rho e^{-\rho t} U(x(t), \dot{x}(t)) dt$$

Taking its partial derivative of this function with respect to ρ :

$$\frac{\partial v}{\partial \rho}(\rho, \chi, \dot{\chi}) = (1 - \rho^2) \int_0^{+\infty} e^{-\rho t} U(x(t), \dot{x}(t)) dt$$

The determination of the optimal discount ρ^* is then independent of the nature of the optimal $\chi^* = (x^*(t))_{t \in \mathbb{R}_+}$. Otherwise stated:

$$\min_{\rho \in [\underline{\rho}, \bar{\rho}]} v(\rho, \chi, \dot{\chi}) = v(\rho^*, \chi, \dot{\chi}) \quad \text{where } \rho^* \in [\underline{\rho}, \bar{\rho}]$$

that implies that the continuous time maximin problem (CTMM) boils down to a fixed discount Ramsey problem $\mathcal{R}(\rho^*)$ with $\rho^* \in [\underline{\rho}, \bar{\rho}]$.

6. COMPARISON WITH THE LITERATURE

6.1 OPTIMIZATION & DYNAMIC PROGRAMMING

GEOFFARD [6]. The closest article from the current contribution is the one of Geoffard [6] that suggests an alternative representation of recursive utility and,

more generally, introduces a class of optimization programs based upon variational utility for which a minimization is to be completed over the set of discount rates. The argument being settled in continuous time and motivated by the worst-case scenario analysis, Geoffard [6] proposes the following *variational utility* model:

$$\min_{r(t) \in \mathcal{R}} \int_{t=0}^{+\infty} f[c(t), B(t), r(t)] dt$$

for \mathcal{R} that denotes the set of admissible paths of discount rates, and where f features current felicity as a function of the current value of consumption $c(t)$, the discount factor $B(t)$ and the discount rate $r(t)$. Under multiplicative separability and for $f[c(t), B(t), r(t)] = B(t)f[c(t), r(t)]$, this formulation recovers a recursive formulation and the intertemporal utility associated to the Bellman-like equation W would currently restate as:

$$\min_{B_t \in \mathcal{B}} \sum_{t=0}^{+\infty} B_t F(c_t, \delta),$$

for $F(c_t, \delta) = (1 - \delta)U(c_t)$ and where \mathcal{B} is a class of admissible discount-factor processes B . Letting however the production be summarized by a function π that is a quasi-concave function of (x, \dot{x}) , the optimization problem considered in Geoffard [1996] lists as:

$$\max_{x \in X} \min_{B \in \mathcal{B}} \int_{t=0}^{+\infty} f[\pi(x, \dot{x}), B, \dot{B}].$$

In opposition to the structure analyzed through the problem (MM) in sections 2-6, the whole analysis is completed under a convexity assumption of the integrand with respect to (B, \dot{B}) , so that the supinf is an infsup problems and equivalent and the order can be reversed. More fundamentally, the separable additive case appears in his approach as a degenerate version of his problem for which the optimal discount rate is trivially fixed at the lower bound of the interval $[\underline{\delta}, \bar{\delta}]$ but doesn't recover the interesting property $(1 - \delta) \sum_{t=0}^{+\infty} \delta^t = 1$ that is specific to the problem (MM) or to its continuous time version (MM) in Section 5.

CLASSICAL THEORY OF CONCAVE DYNAMIC PROGRAMMING Owing to the range of assumptions that have been retained through T1-T5 and V1-V5, the argument naturally compares with the classical arguments of concave program-

ming as listed in sections 4.2 and 4.4 of Stokey, Lucas & Prescott [13]. The uniqueness of the policy function is currently obtained as a result on the convexity assumption T.2 on Γ and concavity assumption V.2 on $V(\cdot, \cdot)$.

More precisely, a comparison between a solution to the optimization problem and a solution to the functional equation may start by noticing that, for continuous bounded objectives as in Section 3 and given Γ, V, x_0 and a fixed δ , up to the existence to a fixed discount Ramsey problem $\mathcal{R}(\delta)$, one can infer from the value function J the optimal sequence. Indeed, for $J(x_0) = v(\chi^*)$, one has $J(x_0) = V(x_0, x_1^*) + \delta J(x_1^*)$ or, more generally

$$J(x_t^*) = V(x_t, x_{t+1}^*) + \delta J(x_{t+1}^*)$$

for every $t = 0, 1, 2, \dots$, the reverse implication being also true for bounded continuous objectives. In this article, Proposition 3 and Theorem 1 provide a related range of results for a problem (MM).

On separate grounds and building upon a taste-dependent representation of inter-temporal utility $\sum_{t=1}^{+\infty} \delta^t u(c_t, c_{t+1})$, Mitra & Nishimura [11] provide a comprehensive analysis of a related reduced form problem with, for $\Phi_{u,F}(x_t, x_{t+1}, x_{t+2}) = u[F(x_t, 1) - x_{t+1}, F(x_{t+1}, 1) - x_{t+2}]$, a maximand $\sum_{t=0}^{\infty} \delta^t \Phi_{u,F}(x_t, x_{t+1}, x_{t+2})$. Interestingly, and along some of the facets of this article, their characterization is univocally rooted on a supermodularity assumption on $\Phi_{u,F}(\cdot, \cdot, \cdot)$ that in turn relates to positive cross derivatives for $\Phi_{u,F}(\cdot, \cdot, \cdot)$, a configuration that results in a monotone increasing policy function.

MARIMON & WERNER (2016) : The paper of Marimon & Werner considers the differentiability of the value function, and the resulting link between the Euler equations and the Bellman equation. This gives rise to the consideration of a *maximin* problem where the minimization criterion is taken on a convex set of multipliers. In contradistinction with their approach, this paper does not consider a saddle-point configuration in a convex-concave problem but rather a *sup inf* criterion in the Bellman equation where the minimization criterion is not convex with respect to the discount factor δ .

6.2 DECISION THEORY

HALEVY [9]. A recent contribution due to Halevy [9] enlightens the subtleties of stationarity, invariance and consistency in temporal preferences. More precisely and for some payoffs ξ and ψ ,

1. \succeq_t is *stationary* if, for every $t, t' \geq 0$, $\xi, \psi \in \mathbb{R}, \Delta_2, \Delta_1 \geq 0$:

$$(\xi, t + \Delta) \sim_t (\psi, t + \Delta) \iff (\xi, t' + \Delta) \sim_{t'} (\psi, t' + \Delta).$$

2. $\{\succeq_t\}_{t=0}^{+\infty}$ are *time invariant* if, for every $t, t' \geq 0$, $\xi, \psi \in \mathbb{R}, \Delta_2, \Delta_1 \geq 0$:

$$(\xi, t + \Delta_1) \sim_t (\psi, t + \Delta_2) \iff (\xi, t' + \Delta_1) \sim_{t'} (\psi, t' + \Delta_2)$$

3. $\{\succeq_t\}_{t=0}^{+\infty}$ exhibit *time consistency* if, for every $t, t' \geq 0$, $\xi, \psi \in \mathbb{R}, \Delta_2, \Delta_1 \geq 0$:

$$(\xi, t + \Delta_1) \sim_t (\psi, t + \Delta_2) \iff (\xi, t + \Delta_1) \sim_{t'} (\psi, t + \Delta_2).$$

The articulation of the class of preferences considered in (MM) with this characterization is then enlightening. Even though they are neither stationary, time-invariant or time-consistent *per se*, the current analysis illustrates how the reference to dynamic programming techniques and the construction of a stationary optimal policy circumvent all these difficulties and allow for determining an explicitly time-consistent policy rule : it is indeed only under the existence of a unique δ^* for a given sequence χ_t that *time consistency* is currently available.

CHAMBERS & ECHENIQUE [2016]. Chambers and Echenique [4] have independently clarified a precise axiomatic basis that underlies a maximin representation closely related to (MM). More precisely, for ξ a sequence and θ a constant sequence, and to a standard to a list of axioms, namely *Monotonicity*, *d-Monotonicity*, *c-Convexity*, *Translation invariance*, *c-Translation invariance*, *Homotheticity* and *Convexity*, they add three new axioms:

1. *Stationarity [New formulation]*: For every $t \in \mathbb{N}$ and every $\lambda \in [0, 1]$, $\xi \geq \theta$ if and only if $\lambda\xi + (1 - \lambda)(\theta, \dots, \theta, \xi) \geq \theta$.
2. *Compensation*: For every t , there are scalars $\underline{\theta}^t$, θ^t and $\bar{\theta}^t$ with $\underline{\theta}^t < \theta^t < \bar{\theta}^t$ such that $(\underline{\theta}^t, \dots, \underline{\theta}^t, \bar{\theta}^t, \dots) \geq \theta^t$.

3. *Continuity at infinity*: For every $\xi \in X$ and for every θ , if $\theta \geq (x_0, \dots, x_T, 0, \dots)$ for every T , then $\theta \geq \xi$.

Their Theorem 4 then asserts that *Under the above list of axioms, the preference relation defined on utility sequences is represented by a criterion (MM)*. Note however that no concavity assumption is retained in their approach while it is currently essential for the establishment of the single-valuedness of the policy correspondence, the argument being based upon the monotonicity of the optimal sequence. More generally, the current argument can be understood as the building minimal structure on their representation so that it would satisfy a stationary dynamic programming argument.

WAKAI [16, 17]. Under axioms of *Conditional Preference, Weak Order, Continuity, Non-Degeneracy, Constant-Independence, Dynamic Consistency, Best-Worst and Stationarity*, Wakai [16, 17] emphasizes, for for $\chi_0 = (x_0, x_1, \dots)$, $\chi_1 = (x_1, x_2, \dots)$ and an affine V , the emergence of a functional $\hat{v}(\cdot)$ that satisfies

$$\hat{v}(\chi_0) = \inf_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta\hat{v}(\chi_1)].$$

He further proves the existence of a convex set \mathbb{D} of positive sequences $\{b_t\}_{t=0}^{\infty}$ such that $\sum_{t=0}^{\infty} b_t = 1$, satisfying

$$\begin{aligned} \hat{v}(\chi_0) &= \inf_{b \in \mathbb{D}} \sum_{t=0}^{\infty} b_t V(x_t, x_{t+1}) \\ &= \inf_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta\hat{v}(\chi_1)]. \end{aligned}$$

This representation is closely related to the ones that are associated with a criterion (MM). First note that the technical difficulties that spring from a set of discount rates $\{(1 - \delta)\delta^t\}_{t=0}^{\infty}$, with $\delta \in \mathcal{D}$ that is not convex are however not explicitly dealt with and that no stationary dynamic programming argument was further developed by this author. Further remark that, in contradistinction with the argument of Wakai [16, 17] and as this was mentioned in Section 3 and as a result of the non-convex dimension in the infimum problem in (MM), the obtention of $\hat{v}(\chi_0) = \{(1 - \delta)V(x_0, x_1) + \delta\hat{v}(\chi_1)\}$ is generically out of reach for the current analysis.

HYPERBOLIC DISCOUNTING. An alternative, and more common representation, relates to the so-called *hyperbolic discounting* or *present-biased* literature. Given an agent, the associated *self* or *incarnation* at time $t \in \mathbb{N}$ ranks consumption sequences according to:

$$u(c_t) + \beta \left[\sum_{\tau=1}^{+\infty} \delta^\tau u(c_{t+\tau}) \right], \delta \in]0, 1[, \quad (3)$$

recent axiomatizations having been put forth by Olea & Strzalecki [12] and Galperti & Strulovici [8]. The associated optimization problem is however not time-consistent. Its solving proceeds from the one of a strategic game between the successive selves rather than from a classical optimization problem. Stationary dynamic programming techniques are then not directly applicable for $\beta \neq 1$.

7. EXAMPLES

7.1 A LOGARITHMIC UTILITY FUNCTION EXAMPLE

Consider an economy with a logarithmic utility function $u(c) = \ln c$ and a one good production technology given by $F(x) = x^\alpha$, where $0 < \alpha < 1$. The payoff function has the following form:

$$V(x, y) = \ln(x^\alpha - y)$$

and the technology is defined by the correspondence:

$$\Gamma(x) = [0, x^\alpha]$$

Given $x_0 > 0$, consider the following maximin problem

$$\sup_{\chi \in \Pi(x_0)} \inf_{\delta \in [\underline{\delta}, \bar{\delta}]} \left[(1 - \delta) \sum_{t=0}^{\infty} \delta^t \ln(x_t^\alpha - x_{t+1}) \right] \quad (\text{MM-ln})$$

That all of assumptions **V1, V3-V5, T1-T5** are satisfied can be checked. Further denoting by $J(x_0)$ the value function of this problem, it derives that:

$$\begin{aligned} J(x_0) &= \max_{\chi \in \Pi(x_0)} \min_{\delta \in [\underline{\delta}, \bar{\delta}]} \left[(1 - \delta) \sum_{t=0}^{\infty} \delta^t \ln(x_t^\alpha - x_{t+1}) \right] \\ &= \min_{\delta \in [\underline{\delta}, \bar{\delta}]} \max_{\chi \in \Pi(x_0)} \left[(1 - \delta) \sum_{t=0}^{\infty} \delta^t \ln(x_t^\alpha - x_{t+1}) \right] \end{aligned}$$

For each $\delta \in [\underline{\delta}, \bar{\delta}]$:

$$\begin{aligned} v_{\max}(\delta, x_0) &:= \max_{x \in \Pi(x_0)} \left[(1 - \delta) \sum_{t=0}^{\infty} \delta^t \ln(x_t^\alpha - x_{t+1}) \right] \\ &= \ln(1 - \alpha\delta) + \frac{\alpha\delta}{1 - \alpha\delta} \ln(\alpha\delta) + \frac{\alpha(1 - \delta)}{1 - \alpha\delta} \ln x_0 \end{aligned} \quad (4)$$

and the optimal sequence $\mathbf{x}_\delta = (x_{\delta,t})_{t=0}^{\infty}$ of the maximum problem (4) is defined by

$$\begin{aligned} x_{\delta,t} &= \alpha\delta(x_{\delta,t-1})^\alpha \\ &= (\alpha\delta)^\alpha x_0^{\alpha(\alpha-1)} x_{\delta,t-1}. \end{aligned}$$

It derives that:

$$J(x_0) = \min_{\delta} v_{\max}(\delta, x_0)$$

It can be checked that

$$\frac{\partial v_{\max}}{\partial \delta} = \frac{\alpha[\ln(\alpha\delta) - (1 - \alpha)\ln x_0]}{(1 - \alpha\delta)^2}$$

and

$$\frac{\partial v_{\max}}{\partial \delta} = 0 \iff x_0 = (\alpha\delta)^{\frac{1}{1-\alpha}}$$

Three cases are to be considered:

- Case 1. $x_0 < (\alpha\underline{\delta})^{1/(1-\alpha)}$.

Then $\partial v_{\max}/\partial \delta > 0$ for all $\delta \in [\underline{\delta}, \bar{\delta}]$. Hence

$$J(x_0) = \min_{\delta} v_{\max}(\delta, x_0) = v_{\max}(\underline{\delta}, x_0).$$

In this case, the optimal sequence $\mathbf{x}^* = (x_t^*)_{t=0}^{\infty}$ is described by

$$x_t^* = \lambda x_{t-1}^*, \quad \text{where } \lambda = (\alpha\underline{\delta})^\alpha x_0^{\alpha(\alpha-1)} > 1$$

then \mathbf{x}^* is strictly increasing and converges to $(\alpha\underline{\delta})^{\frac{1}{1-\alpha}}$.

- Case 2. $x_0 > (\alpha\bar{\delta})^{1/(1-\alpha)}$. Then $\partial v_{\max}/\partial \delta < 0$ for all $\delta \in [\underline{\delta}, \bar{\delta}]$. Hence

$$J(x_0) = \min_{\delta} v_{\max}(\delta, x_0) = v_{\max}(\bar{\delta}, x_0).$$

In this case, the optimal sequence $\mathbf{x}^* = \{x_t^*\}_{t=0}^{\infty}$ is described by

$$x_t^* = \lambda x_{t-1}^*, \quad \text{where } \lambda = (\alpha\bar{\delta})^\alpha x_0^{\alpha(\alpha-1)} < 1$$

then \mathbf{x}^* is strictly decreasing and converges to $(\alpha\bar{\delta})^{1/(1-\alpha)}$.

- Case 3. $x_o \in [(\alpha\underline{\delta})^{1/(1-\alpha)}, (\alpha\bar{\delta})^{1/(1-\alpha)}]$. It derives that:

$$\frac{\partial z}{\partial \delta} = 0 \quad \Leftrightarrow \quad \delta = \frac{x_o^{1-\alpha}}{\alpha} := \delta_o$$

and

$$J(x_o) = \min_{\delta} z(\delta, x_o) = z(\delta_o, x_o)$$

In this case $x_t^* = \alpha\delta_o(x_{t-1}^*)^\alpha = x_o$ for all $t \geq 0$.

Even though this characterization more closely corresponds to the unbounded from below case analyzed in Section 4, the results directly replicate the ones listed in Proposition 3.

7.2 A MODEL WITHOUT SUPER-MODULARITY

Consider now a model without super-modularity but for which the property $W = J$ is preserved. The proof builds upon the monotonicity of the optimal policy solution of the fixed discount Ramsey problem.

In Boldrin & Deneckere [3] economy with two sectors, the payoff function has the form

$$V(x, y) = (1 - y)^\alpha (x - \gamma y)^{1-\alpha}, \quad \alpha \in]0, 1[; \gamma \in]0, 1[$$

and the technology correspondence states as:

$$\Gamma(x) = \left[0, \min \left\{ 1, \frac{x}{\gamma} \right\} \right]$$

Given $x_o \in [0, 1]$, the maximin problem is available as:

$$\max_{x \in \Pi(x_o)} \min_{\delta \in [\underline{\delta}, \bar{\delta}]} \left[(1 - \delta) \sum_{t=0}^{\infty} \delta^t (1 - x_{t+1})^\alpha (x_t - \gamma x_{t+1})^{1-\alpha} \right] \quad (\text{MM-Boldrin})$$

The partial and second-order cross derivatives of V derive as:

$$\begin{aligned} V_1(x, y) &= (1 - \alpha) \frac{(1 - y)^\alpha}{(x - \gamma y)^\alpha}, \\ V_2(x, y) &= -\frac{\alpha(x - \gamma) + \gamma(1 - y)}{(1 - y)^{1-\alpha} (x - \gamma y)^\alpha}, \\ V_{12}(x, y) &= \frac{\alpha(1 - \alpha)}{(1 - y)^{1-\alpha} (x - \gamma y)^{1+\alpha}} (\gamma - x). \end{aligned}$$

First assume that every value in \mathcal{D} satisfies the condition of Proposition 1 in Boldrin & Deneckere [3], namely:

$$\gamma < \delta < \frac{\gamma}{1-\alpha} \quad \text{for any } \delta \in \mathcal{D}.$$

Observe that $V_{12}(x, y) > 0$ if $x < \gamma$ but $V_{12}(x, y) < 0$ if $x > \gamma$. Under this conjunction, for each δ , the fixed discount Ramsey problem $\mathcal{R}(\delta)$ has an optimal policy function φ_δ which is strictly increasing over $[0, \gamma]$ and strictly decreasing over $[\gamma, 1]$. This function has a unique non trivial fix point $x^{\delta*}$ that is determined by

$$x^{\delta*} = \frac{(\delta - \gamma)(1 - \alpha)}{(\delta - \gamma)(1 - \alpha) + \alpha(1 - \gamma)}. \quad (5)$$

Observe that $x^{\delta*}$ is increasing in respect to δ and for all $\underline{\delta} \leq \delta \leq \bar{\delta}$, $x^{\underline{\delta}*} \leq x^{\delta*} \leq x^{\bar{\delta}*}$.

The aim will now be to prove that for all $\delta \in \mathcal{D}$, the solution of the fixed discount Ramsey problem $\mathcal{R}(\delta)$ is monotonic.

Since the inequality $\gamma < \delta < \gamma/(1 - \alpha)$ has been assumed to prevail for every $\delta \in \mathcal{D}$, it is obtained that $x^{\delta*} < \gamma$. Since φ_δ is increasing over $[0, \gamma]$ and $\varphi_\delta(\gamma) < \gamma$, for any $x_0 \leq \gamma$, the sequence defined by $x_t = \varphi_\delta^t(x_0)$ is monotonic and converges to $x^{\delta*}$. Using the same technique as in the proof of Propositions 2 and 3, it can further be proved that $\{x_t\}_{t=0}^\infty$ also satisfies

$$W(x_t) = \min_{\delta \in \mathcal{D}} \left\{ (1 - \delta)V(x_t, x_{t+1}) + \delta W(x_{t+1}) \right\},$$

for any $t \geq 0$ and

$$W(x_0) = J(x_0).$$

Suppose that $\gamma < x_0 \leq 1$. The aim is to prove that $\{x_t\}_{t=0}^\infty$ is decreasing. This is equivalent to prove that $x^{\delta*} < x_1 < \gamma$. Indeed, if $x_1 < x^{\delta*}$, $x_2 > x_1$ obtains and the sequence is not monotonic. If $x_1 > \gamma$, since $V_{12}(x, y) < 0$ for $x > \gamma$ and $x_1 < x_0$, it is obtained that $\varphi_\delta(x_1) > \varphi_\delta(x_0)$ or $x_2 > x_1$, and the sequence is not monotonic.

Since φ_δ attains its maximum at γ , for any $\gamma < x_0 < 1$ it derives that $\varphi_\delta(x_0) \leq \varphi_\delta(\gamma) < \gamma$. It will now be proved that $x^{\delta*} < \varphi_\delta(x_0)$ for any $\gamma \leq x_0 \leq 1$. Since on this interval φ_δ is decreasing, the inequality is equivalent to $x^{\delta*} < \varphi_\delta(1)$, with $\varphi_\delta(1)$ the solution to

$$V_2(1, x) + \delta W'_\delta(x) = 0.$$

Since the function $V_2(1, x) + \delta W'_\delta(x)$ is strictly decreasing, $x^{\delta^*} < \varphi_\delta(1)$ if and only if

$$V_2(1, x^{\delta^*}) + \delta W'_\delta(x^{\delta^*}) > 0. \quad (6)$$

From the optimality property

$$V_2(x^{\delta^*}, x^{\delta^*}) + \delta W'_\delta(x^{\delta^*}) = 0,$$

hence the inequation (6) is equivalent to

$$V_2(1, x^{\delta^*}) > V_2(x^{\delta^*}, x^{\delta^*}).$$

This inequality is equivalent to

$$\frac{\alpha(1-\gamma) + \gamma(1-x^{\delta^*})}{(1-\gamma x^{\delta^*})^\alpha} < \frac{\alpha(x^{\delta^*}-\gamma) + \gamma(1-x^{\delta^*})}{(x^{\delta^*}-\gamma x^{\delta^*})^\alpha}. \quad (7)$$

By substituting x^{δ^*} from (5) to (7), one can easily check that the inequality (7) is equivalent to

$$\frac{\alpha + \delta - \alpha\delta}{[(\delta - \gamma)(1 - \alpha) + \alpha]^\alpha} < \frac{\delta - \alpha\delta}{[(\delta - \gamma)(1 - \alpha)]^\alpha} \iff 1 + \frac{\alpha}{\delta(1 - \alpha)} < \left[1 + \frac{\alpha}{(\delta - \gamma)(1 - \alpha)}\right]^\alpha \quad (8)$$

From the assumption of the set \mathcal{D} :

$$\delta < \frac{\gamma}{1 - \alpha} \implies \delta - \alpha\delta < \gamma \implies \frac{\alpha}{\delta - \gamma} > \frac{1}{\delta}$$

This implies

$$\begin{aligned} \left[1 + \frac{\alpha}{(\delta - \gamma)(1 - \alpha)}\right]^\alpha &> \left[1 + \frac{1}{\delta(1 - \alpha)}\right]^\alpha \\ &> 1 + \frac{\alpha}{\delta(1 - \alpha)} \end{aligned}$$

that makes true the inequality (8).

Using the monotonicity of the solution of the fixed discount Ramsey problems, it derives that, for any x_0 , $W(x_0) = J(x_0)$ and the decision rule of the maximin problem is stationary. Moreover, similarly to Proposition 3, for $x_0 < x^{\underline{\delta}}$, the discount factor chosen is $\underline{\delta}$, and optimal sequence converges to $x_{\underline{\delta}}$, for $x_0 > x^{\bar{\delta}}$, the discount factor chosen is $\bar{\delta}$, and optimal sequence converges to $x_{\bar{\delta}}$, or $x^{\underline{\delta}} \leq x_0 \leq x^{\bar{\delta}}$, the optimal sequence is constant, and the chosen discount factor satisfies $x^\delta = x_0$. The detailed proof is given in the Appendix.

A. PROOF OF LEMMA 1

Proof. Obviously, for any $\tilde{\chi} \in \Pi(x_0)$:

$$\inf_{\delta \in \mathcal{D}} v(\delta, \tilde{\chi}) \leq \inf_{\delta \in \mathcal{D}} \sup_{\chi \in \Pi(x_0)} v(\delta, \chi).$$

This implies

$$\sup_{\tilde{\chi} \in \Pi(x_0)} \inf_{\delta \in \mathcal{D}} v(\delta, \tilde{\chi}) \leq \inf_{\delta \in \mathcal{D}} \sup_{\chi \in \Pi(x_0)} v(\delta, \chi).$$

and the statement follows. QED

B. PROOF OF PROPOSITION 1

Proof. (i) This is a straightforward application of the Weierstrass Theorem.

(ii) a/ Fix $\chi_1, \chi_2 \in \Pi(x_0)$ such that $\chi_1 \neq \chi_2$ and $v(\chi_1, \delta), v(\chi_2, \delta) > -\infty$ for every $\delta \in [\underline{\delta}, \bar{\delta}]$. Fix $0 < \lambda < 1$. Denote by $\chi_\lambda = (1 - \lambda)\chi_1 + \lambda\chi_2$. From the convexity of Γ , it derives that $\chi \in \Pi(x_0)$. Fix $\delta_\lambda \in \mathcal{D}$ such that $\hat{v}(\chi_\lambda) = v(\delta_\lambda, \chi_\lambda)$.

From the strict concavity of V , it is obtained that

$$\begin{aligned} \hat{v}(\chi_\lambda) &= v(\delta_\lambda, (1 - \lambda)\chi_1 + \lambda\chi_2) \\ &> (1 - \lambda)v(\delta_\lambda, \chi_1) + (1 - \lambda)v(\delta_\lambda, \chi_2) \\ &\geq (1 - \lambda) \min_{\delta \in \mathcal{D}} v(\delta, \chi_1) + \lambda \min_{\delta \in \mathcal{D}} v(\delta, \chi_2) \\ &= (1 - \lambda)\hat{v}(\chi_1) + \lambda\hat{v}(\chi_2). \end{aligned}$$

(iii) b/ From (1), take the sequence χ^n such that $\lim_{n \rightarrow \infty} \hat{v}(\chi^n) = J(x_0)$. Since from assumption T2, $\Pi(x_0)$ is compact in the product topology, it can be supposed that $\chi^n \rightarrow \chi^*$. It will first be proved that $\hat{v}(\chi^*) > -\infty$.

Fix any $\epsilon > 0$, there exists N big enough such that for $n \geq N$, $v(\chi^n) > J(x_0) - \epsilon$ prevails. Observe that from assumption T2, there exists $C > 0$ such that $0 \leq x_t \leq C$ for any $\chi \in \Pi(x_0)$. Hence there exists T satisfying for any $\chi \in \Pi(x_0)$, for any $\delta \in \mathcal{D}$,

$$(1 - \delta) \sum_{t=T+1}^{\infty} \delta^t V(x_t, x_{t+1}) < \epsilon.$$

For any $n \geq N$, for any $\delta \in \mathcal{D}$:

$$J(x_0) - \epsilon \leq v(\chi^n)$$

$$\leq (1 - \delta) \sum_{t=0}^T \delta^t V(x_t^n, x_{t+1}^n) + (1 - \delta) \sum_{t=T+1}^{\infty} \delta^t V(x_t^n, x_{t+1}^n),$$

which implies

$$\begin{aligned} (1 - \delta) \sum_{t=0}^T \delta^t V(x_t^n, x_{t+1}^n) &\geq J(x_0) - \epsilon - (1 - \delta) \sum_{t=T+1}^{\infty} \delta^t V(x_t^n, x_{t+1}^n) \\ &\geq J(x_0) - 2\epsilon. \end{aligned}$$

Letting n converge to infinity, for any $\delta \in \mathcal{D}$,

$$(1 - \delta) \sum_{t=0}^T \delta^t V(x_t^*, x_{t+1}^*) \geq J(x_0) - 2\epsilon.$$

This inequality is true for any T big enough. Letting T tends to infinity, for any $\delta \in \mathcal{D}$:

$$(1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t^*, x_{t+1}^*) \geq J(x_0) - 2\epsilon.$$

Hence $\hat{v}(\chi^*) \geq J(x_0) - 2\epsilon$.

The parameter ϵ being chosen arbitrarily, $\hat{v}(\chi^*) \geq J(x_0)$. From the strict concavity of \hat{v} , χ^* is the unique solution of $\sup_{\chi \in \Pi(x_0)} \hat{v}(\chi)$. QED

C. PROOF OF LEMMA 2

Proof. From the compactness of \mathcal{D} and $\Gamma(x_0)$ and for every $x_0 > 0$, let T denote the operator from the space of continuous function into itself:

$$Tg(x_0) = \max_{x_1 \in \Gamma(x_0)} \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta g(x_1)].$$

Supposing that $g(x) \leq \tilde{g}(x)$ for any x , $Tg(x) \leq T\tilde{g}(x)$ for any x . Indeed, for each $x_1 \in \Gamma(x_0)$, for every δ , it is obtained that:

$$(1 - \delta)V(x_0, x_1) + \delta g(x_1) \leq (1 - \delta)V(x_0, x_1) + \delta \tilde{g}(x_1),$$

which implies

$$\min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta g(x_1)] \leq \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta \tilde{g}(x_1)].$$

The inequality being obviously true for every x_1 , $Tv(x_0) \leq T\tilde{g}(x_0)$ in turn prevails.

By the same arguments, it can be proved that for every constant $a > 0$, $T(g + a)(x_0) \leq Tg(x_0) + \bar{\delta}a$.

Since T satisfies the properties of Blackwell, i.e, monotonicity and discounting, by Theorem 3.3 in Stokey, Lucas & Prescott [13], the operator T is a contracting map and has a unique fixed point.

In preparation for the concavity of this fixed point function, it will now be proved that the concavity of g implies the concavity of Tg . Fix x_0 and y_0 and fix (δ_x, x_1) and (δ_y, y_1) such that

$$\begin{aligned} Tg(x_0) &= (1 - \delta_x)V(x_0, x_1) + \delta_x g(x_1) \\ &= \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta g(x_1)], \\ Tg(y_0) &= (1 - \delta_y)V(y_0, y_1) + \delta_y g(y_1) \\ &= \min_{\delta \in \mathcal{D}} [(1 - \delta)V(y_0, y_1) + \delta g(y_1)]. \end{aligned}$$

For any $0 \leq \lambda \leq 1$, define $x^\lambda = (1 - \lambda)x_0 + \lambda y_0$, $x_1^\lambda = (1 - \lambda)x_1 + \lambda y_1$. For any $\delta \in \mathcal{D}$, it derives that:

$$\begin{aligned} (1 - \delta_x)V(x_0, x_1) + \delta_x g(x_1) &\leq (1 - \delta)V(x_0, x_1) + \delta g(x_1), \\ (1 - \delta_y)V(y_0, y_1) + \delta_y g(y_1) &\leq (1 - \delta)V(y_0, y_1) + \delta g(y_1). \end{aligned}$$

Hence, for any $\underline{\delta} \leq \delta \leq \bar{\delta}$:

$$\begin{aligned} (1 - \lambda)Tg(x_0) + \lambda Tg(y_0) &= (1 - \lambda)((1 - \delta_x)V(x_0, x_1) + \delta_x g(x_1)) \\ &\quad + \lambda((1 - \delta_y)V(y_0, y_1) + \delta_y g(y_1)) \\ &\leq (1 - \lambda)[(1 - \delta)V(x_0, x_1) + \delta g(x_1)] \\ &\quad + \lambda[(1 - \delta)V(y_0, y_1) + \delta g(y_1)] \\ &\leq (1 - \delta)V(x_0^\lambda, x_1^\lambda) + \delta g(x_1^\lambda). \end{aligned}$$

The inequality being true for every δ , it is obtained that:

$$\begin{aligned} (1 - \lambda)Tg(x_0) + \lambda Tg(x'_0) &\leq \min_{\delta \in \mathcal{D}} ((1 - \delta)V(x_0^\lambda, x_1^\lambda) + \delta g(x_1^\lambda)) \\ &\leq Tg(x_0^\lambda). \end{aligned}$$

Take $g^0(x_0) = 0$ for every x_0 , define $g^{n+1}(x_0) = Tg^n(x_0)$ for any $n \geq 0$. By induction, $g^n(x_0)$ is a concave function for any n . Taking limits, $W(x_0) = \lim_{n \rightarrow \infty} g^n(x_0) = \lim_{n \rightarrow \infty} T^n g^0(x_0)$, the concavity property is available for W .

The strict concavity of W is now going to be established. Consider $x_o \neq y_o$. Define $(\delta_x, x_1), (\delta_y, y_1), x_o^\lambda, x_1^\lambda$ as in the first part of this proof. Using exactly the same arguments and calculus and the strictly concavity of V , for any $\delta \in \mathcal{D}$, it is obtained that:

$$\begin{aligned} (1 - \lambda)W(x_o) + \lambda W(x_1) &\leq (1 - \delta)((1 - \lambda)V(x_o, x_1) + \lambda V(y_o, y_1)) \\ &\quad + \delta((1 - \lambda)W(x_1) + \lambda W(y_1)) \\ &< (1 - \delta)V(x_o^\lambda, x_1^\lambda) + \delta W(x_1^\lambda). \end{aligned}$$

From the compacity of the set of discount factors \mathcal{D} , it derives that:

$$\begin{aligned} (1 - \lambda)W(x_o) + \lambda W(x_1) &< \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_o^\lambda, x_1^\lambda) + \delta W(x_1^\lambda)] \\ &\leq W(x_o^\lambda). \end{aligned}$$

QED

D. PROOF OF PROPOSITION 2

Proof. Define $w(x_o, x_1) = \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_o, x_1) + \delta W(x_1)]$.

(i) Observe that the function $w(x_o, x_1)$ is strictly concave in x_1 . Indeed, for any $x_1 \neq y_1$ belonging to $\Gamma(x_o)$, and $0 < \lambda < 1$, define $x_1^\lambda = (1 - \lambda)x_1 + \lambda y_1$. For any $\delta \in \mathcal{D}$:

$$\begin{aligned} (1 - \lambda)w(x_o, x_1) + \lambda w(x_o, y_1) &\leq (1 - \lambda)[(1 - \delta)V(x_o, x_1) + \delta W(x_1)] \\ &\quad + \lambda[(1 - \delta)V(x_o, y_1) + \delta W(y_1)] \\ &< (1 - \delta)V(x_o, x_1^\lambda) + \delta W(x_1^\lambda). \end{aligned}$$

The definition set for discount factors δ being compact, it is obtained that:

$$\begin{aligned} (1 - \lambda)w(x_o, x_1) + \lambda w(x_o, y_1) &< \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_o, x_1^\lambda) + \delta W(x_1^\lambda)] \\ &= w(x_o, x_1^\lambda). \end{aligned}$$

The strict concavity of $w(x_o, \cdot)$ implies that x_1 which maximizes this function is unique.

(ii) The monotonicity of the policy function φ is now going to be established. For that purpose, it must first be checked that there cannot exist a couple (x_o, z_o) such that $x_o < z_o$ and $\varphi(x_o) > \varphi(z_o)$. Suppose the opposite and let $y_o = \operatorname{argmin}_{[x_o, z_o]} \varphi(y)$.

(a) Consider the first case, $y_o < \varphi(y_o)$. This implies $W(y_o) < W(\varphi(y_o))$. Take

$$\delta_{y_o} \in \operatorname{argmin}_{\delta \in \mathcal{D}} [(1 - \delta)V(y_o, \varphi(y_o)) + \delta W(\varphi(y_o))].$$

It derives that:

$$(1 - \delta_{y_o})V(y_o, \varphi(y_o)) + \delta_{y_o} W(\varphi(y_o)) < W(\varphi(y_o)),$$

that implies $V(y_o, \varphi(y_o)) < W(\varphi(y_o))$.

There exists $\epsilon > 0$ such that

$$V(y, \varphi(y)) < W(\varphi(y)) \text{ for any } y_o - \epsilon < y < y_o + \epsilon.$$

For every y belonging to this interval, the following is satisfied:

$$\operatorname{argmin}_{\delta \in \mathcal{D}} [(1 - \delta)V(y, \varphi(y)) + \delta W(\varphi(y))] = \{\underline{\delta}\}.$$

Recall that, in this case, for any $y_o - \epsilon < y < y_o + \epsilon$

$$\varphi(y) = \operatorname{argmax}_{y' \in \Gamma(y)} [(1 - \underline{\delta})V(y, y') + \underline{\delta}W(y')].$$

From the super modularity of V , using the Topkis's theorem quoted in Amir [1], φ is strictly increasing (ascending) in $]y_o - \epsilon, y_o + \epsilon[$, hence for $y_o - \epsilon < y < y_o$, the inequality $\varphi(y) < \varphi(y_o)$, a contradiction with the choice of y_o .

(b) The second case, if $y_o > \varphi(y_o)$, is faced by applying the same arguments and a contradiction is similarly obtained.

(c) Consider the third case. $x_o < y_o$ and $\varphi(x_o) > \varphi(y_o) = y_o$ are simultaneously satisfied. But, and from Assumption **V4**, V is increasing in its first argument and decreasing in its second one,

$$\begin{aligned} V(x_o, \varphi(x_o)) &< V(y_o, y_o) \\ &= W(y_o) \\ &< W(\varphi(x_o)). \end{aligned}$$

Hence

$$\operatorname{argmin}_{\delta \in \mathcal{D}} [(1 - \delta)V(x_o, \varphi(x_o)) + \delta W(\varphi(x_o))] = \{\underline{\delta}\}.$$

Since $\varphi(x_0) > x_0$, $\varphi(x) > x$ prevails in a neighborhood of x_0 . Denoting by I_{x_0} the set of $z \in]x_0, y_0[$ such that for any $x_0 < x < z$, it derives that:

$$V(x, \varphi(x)) < W(\varphi(x)) \text{ which is equivalent to } x < \varphi(x).$$

It is further obvious that for any $x_0 < x < z$ with $z \in I_{x_0}$, one has:

$$\underset{\delta \in \mathcal{D}}{\operatorname{argmin}}[(1 - \delta)V(x, \varphi(x)) + \delta W(\varphi(x))] = \{\underline{\delta}\},$$

which implies for any $x_0 < x < z$:

$$\varphi(x) = \underset{x' \in \Gamma(x)}{\operatorname{argmax}}[(1 - \underline{\delta})V(x, x') + \underline{\delta}W(x')].$$

Hence on the interval I_{x_0} , from the super modularity of V , the function φ is strictly increasing.

Take $\bar{z} = \sup(I_{x_0})$. If $z_0 < \varphi(z_0)$, then and by the continuity of V and W , $\varphi(x) > x$ prevails for any $x_0 < x < \bar{z} + \epsilon$, with ϵ sufficiently small: a contradiction. Hence $\varphi(\bar{z}) \leq \bar{z}$. From the continuity of φ , $\varphi(\bar{z}) = \bar{z}$.

By the increasing property of φ on I_{x_0} , it is obtained that:

$$\begin{aligned} \varphi(x_0) &< \varphi(\bar{z}) \\ &\leq z \\ &\leq y_0 \\ &= \varphi(y_0) \\ &< \varphi(x_0), \end{aligned}$$

a contradiction.

It has been proved that, for any $x_0 < z_0$, $\varphi(x_0) \leq \varphi(z_0)$. Suppose that there exists a couple (x_0, z_0) such that $x_0 < z_0$ and $\varphi(x_0) = \varphi(z_0)$. This implies for any $x_0 \leq y \leq z_0$, $\varphi(y) = \varphi(x_0) = \varphi(z_0)$. Hence there exists $y_0 \in [x_0, z_0]$ such that $\varphi(y_0) \neq y_0$. Using the same arguments as in the first part of the proof, φ is to be strictly increasing in a neighborhood of y_0 , that contradicts with the result that φ is constant on $[x_0, z_0]$.

The monotonicity of $\chi^* = \{x_t^*\}_{t=0}^\infty$ is a direct consequence of the increasing property of φ . QED

E. PROOF OF PROPOSITION 3

Proof. First consider the case of a sequence $\chi^* = \{x_t^*\}_{t=0}^\infty = \{\varphi^t(x_0)\}_{t=0}^\infty$ that is strictly increasing. The objective will be to prove that χ^* is solution of a fixed discount Ramsey problem $\mathcal{R}(\underline{\delta})$.

Taking any $\delta^* \in \operatorname{argmin}_{\delta \in \mathcal{D}} [(1-\delta)V(x_0, x_1^*) + \delta W(x_1^*)]$, it derives that $W(x_0) < W(x_1^*)$, hence

$$(1 - \delta^*)V(x_0) + \delta^*W(x_1^*) < W(x_1^*).$$

This implies that $V(x_0, x_1^*) < W(x_1^*)$. Hence

$$\operatorname{argmin}_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1^*) + \delta W(x_1^*)] = \{\underline{\delta}\}.$$

Since $x_0 < x_1^*$, $\varphi(x_0) < \varphi(x_1^*)$, or $x_1^* < x_2^*$. By induction, it is obtained that $x_t^* < x_{t+1}^*$ and

$$\operatorname{argmin}_{\delta \in \mathcal{D}} [(1 - \delta)V(x_t^*, x_{t+1}^*) + \delta W(x_{t+1}^*)] = \{\underline{\delta}\} \text{ for any } t \geq 0.$$

This implies for any finite T

$$\begin{aligned} W(x_0) &= (1 - \underline{\delta})V(x_0, x_1^*) + \underline{\delta}W(x_1^*) \\ &= (1 - \underline{\delta})V(x_0, x_1^*) + (1 - \underline{\delta})\underline{\delta}V(x_1^*, x_2^*) + \underline{\delta}^2 W(x_{t+2}^*) \\ &\dots \\ &= (1 - \underline{\delta}) \sum_{t=0}^T \underline{\delta}^t V(x_t^*, x_{t+1}^*) + \underline{\delta}^{T+1} W(x_{T+1}^*). \end{aligned}$$

Observe that from assumptions **T2** and **T3**, there exists a $C > 0$ such that $x_0 \leq x_t^* < C$.

C. This implies

$$\lim_{T \rightarrow \infty} \underline{\delta}^T W(x_T^*) = 0.$$

Hence

$$\begin{aligned} W(x_0) &= (1 - \underline{\delta}) \sum_{t=0}^T \underline{\delta}^t V(x_t^*, x_{t+1}^*) + \underline{\delta}^{T+1} W(x_{T+1}^*) \\ &= (1 - \underline{\delta}) \sum_{t=0}^{\infty} \underline{\delta}^t V(x_t^*, x_{t+1}^*), \end{aligned}$$

by letting T tends to infinity.

The aim is now to prove that χ^* is a solution of the fixed discount problem $\mathcal{R}(\underline{\delta})$. Suppose the contrary and denote $\hat{\chi} = \{\hat{x}_t\}_{t=0}^{\infty}$ as the unique solution of $\mathcal{R}(\underline{\delta})$. The sequence $\{W(\hat{x}_t)\}$ being uniformly bounded, the date T can be selected as being large enough in order to satisfy:

$$(1 - \underline{\delta}) \sum_{t=0}^T \underline{\delta}^t V(\hat{x}_t, \hat{x}_{t+1}) + \underline{\delta}^{T+1} W(\hat{x}_{T+1}) > (1 - \underline{\delta}) \sum_{t=0}^{\infty} \underline{\delta}^t V(x_t^*, x_{t+1}^*).$$

For $0 \leq \lambda \leq 1$, define $x_t^\lambda = (1 - \lambda)\hat{x}_t + \lambda x_t^*$. Recall that

$$V(x_t^*, x_{t+1}^*) < W(x_{t+1}^*) \text{ for every } t \geq 0.$$

Take λ sufficiently close from 1 so that, for every $0 \leq t \leq T$, the inequality $V(x_t^\lambda, x_{t+1}^\lambda) < W(x_{t+1}^\lambda)$ is satisfied: for any $0 \leq t \leq T$,

$$\underset{\delta \in \mathcal{D}}{\operatorname{argmin}} [(1 - \delta)V(x_t^\lambda, x_{t+1}^\lambda) + \delta W(x_{t+1}^\lambda)] = \{\underline{\delta}\}.$$

Hence

$$\begin{aligned} W(x_0) &\geq (1 - \underline{\delta})V(x_0, x_1^\lambda) + \underline{\delta}W(x_1^\lambda) \\ &\geq (1 - \underline{\delta})V(x_0, x_1^\lambda) + (1 - \underline{\delta})\underline{\delta}V(x_1^\lambda, x_2^\lambda) + \underline{\delta}^2 W(x_2^\lambda) \\ &\dots \\ &\geq (1 - \underline{\delta}) \sum_{t=0}^T \underline{\delta}^t V(x_t^\lambda, x_{t+1}^\lambda) + \underline{\delta}^{T+1} W(x_{T+1}^\lambda). \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{t=0}^T \underline{\delta}^t V(x_t^\lambda, x_{t+1}^\lambda) &\geq (1 - \lambda) \sum_{t=0}^T \underline{\delta}^t V(\hat{x}_t, \hat{x}_{t+1}) + \lambda \sum_{t=0}^T \underline{\delta}^t V(x_t^*, x_{t+1}^*), \\ W(x_{T+1}^\lambda) &\geq (1 - \lambda)W(\hat{x}_{T+1}) + \lambda W(x_{T+1}^*). \end{aligned}$$

Hence, and from the definition of T

$$\begin{aligned} W(x_0) &\geq (1 - \underline{\delta}) \left((1 - \lambda) \sum_{t=0}^T \underline{\delta}^t V(\hat{x}_t, \hat{x}_{t+1}) + \lambda \sum_{t=0}^T \underline{\delta}^t V(x_t^*, x_{t+1}^*) \right) \\ &\quad + \underline{\delta}^{T+1} ((1 - \lambda)W(\hat{x}_{T+1}) + \lambda W(x_{T+1}^*)) \\ &= (1 - \lambda) \left((1 - \underline{\delta}) \sum_{t=0}^T \underline{\delta}^t V(\hat{x}_t, \hat{x}_{t+1}) + \underline{\delta}^{T+1} W(\hat{x}_{T+1}) \right) \end{aligned}$$

$$\begin{aligned}
& + \lambda \left((1 - \underline{\delta}) \sum_{t=0}^T \underline{\delta}^t V(x_t^*, x_{t+1}^*) + \underline{\delta}^{T+1} W(x_{T+1}^*) \right) \\
& > (1 - \underline{\delta}) \sum_{t=0}^{\infty} \underline{\delta}^t V(x_t^*, x_{t+1}^*) \\
& = W(x_0),
\end{aligned}$$

a contradiction.

Observe that since $\chi^* = \operatorname{argmax} \mathcal{R}(\underline{\delta})$, the sequence $\{x_t^*\}_{t=0}^{\infty}$ converges to a steady state of $\mathcal{R}(\underline{\delta})$. Supposing that $x_0 > x_1^*$, by using the same arguments, it can be proved that χ is strictly decreasing, is a solution of a fixed discount problem $\mathcal{R}(\bar{\delta})$ and converges to a steady state of that problem $\mathcal{R}(\bar{\delta})$.

Consider now the case $\varphi(x_0) = x_0$ and hence $x_t^* = x_0$ for any t . There are two cases

- i) For any $\epsilon > 0$, there exists $x_0 - \epsilon < x < x_0 + \epsilon$ satisfying $x \neq \varphi(x)$.
- ii) There exists $\epsilon > 0$ such that for any $x_0 - \epsilon < x < x_0 + \epsilon$, $x = \varphi(x)$.

Consider the case i). Take a sequence x_0^n that converges to x_0 and such that $\varphi(x_0^n) \neq x_0^n$ for any n . Since, in this case, the sequence is either solution of $\mathcal{R}(\underline{\delta})$, or of $\mathcal{R}(\bar{\delta})$, without loss of generality, it can be assumed that, for any n , the sequence $\{\varphi^t(x_0^n)\}_{t=0}^{\infty}$ is a solution of the fixed discount Ramsey problem $\mathcal{R}(\delta)$, with $\delta \in \{\underline{\delta}, \bar{\delta}\}$.

But $\lim_{n \rightarrow \infty} \varphi(x_0^n) = \varphi(x_0) = x_0$, which implies $\lim_{n \rightarrow \infty} \varphi^2(x_0^n) = x_0$. From the Euler equation

$$V_2(x_0^n, \varphi(x_0^n)) + \delta V_1(\varphi(x_0^n), \varphi^2(x_0^n)) = 0,$$

it is derived that:

$$V_2(x_0, x_0) + \delta V_1(x_0, x_0) = 0.$$

Hence the sequence $x_t = x_0$ for any t is solution of $\mathcal{R}(\delta)$. Suppose now that for $x_0 - \epsilon < x < x_0 + \epsilon$, $\varphi(x) = x$. This implies $W(x) = V(x, x)$ for $x \in (x_0 - \epsilon, x_0 + \epsilon)$, that in its turn implies

$$W'(x_0) = V_1(x_0, x_0) + V_2(x_0, x_0).$$

Observe that, the function $(1 - \delta)V(x_0, x_1) + \delta W(x_1)$ is linear in δ and concave in x_1 , the following also prevails:

$$\max_{x_1 \in \Gamma(x_0)} \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_0, x_1) + \delta W(x_1)] = \min_{\delta \in \mathcal{D}} \max_{x_1 \in \Gamma(x_0)} [(1 - \delta)V(x_0, x_1) + \delta W(x_1)].$$

From $\varphi(x_0) = x_0$, there exists $\delta \in \mathcal{D}$ such that

$$(1 - \delta)V_2(x_0, x_0) + \delta W'(x_0) = 0.$$

This implies

$$(1 - \delta)V_2(x_0, x_0) + \delta V_1(x_0, x_0) + \delta V_2(x_0, x_0) = 0,$$

that is equivalent to

$$V_2(x_0, x_0) + \delta V_1(x_0, x_0) = 0,$$

For any t , the sequence $x_t = x_0$ is thus a solution of $\mathcal{R}(\delta)$.

QED

F. PROOF OF LEMMA 3

Proof. Obviously, it is well known in the literature on fixed discount Ramsey problems that W is strictly concave in $]0, x^\delta]$, $[x^\delta, x^{\bar{\delta}}]$ and $[x^{\bar{\delta}}, +\infty[$. We just have to prove that W is differentiable at x^δ and $x^{\bar{\delta}}$.

The left derivative of W at x^δ is equal to the derivative of J_δ :

$$W'_-(x^\delta) = J'_\delta(x^\delta).$$

It is well known that

$$\begin{aligned} J'_\delta(x^\delta) &= (1 - \delta)V_1(x^\delta, x^\delta) \\ &= V_1(x^\delta, x^\delta) + V_2(x^\delta, x^\delta) \\ &= W'_+(x_0). \end{aligned}$$

Hence $W'_-(x^\delta) = W'_+(x^\delta)$, or W is differentiable at x^δ . The proof of the differentiability at $x^{\bar{\delta}}$ makes use of the same arguments. QED

G. PROOF OF THEOREM 1

Proof. (i) This part is a corollary of Proposition 3.

(ii) For each $x_0 > 0$, define $\chi^* = \{x_t^*\}_{t=0}^\infty = \{\varphi^t(x_0)\}_{t=0}^\infty$. Using the Propositions 3, there exists $\delta^* \in \mathcal{D}$ such that χ^* is a solution of $\mathcal{R}(\delta^*)$ and

$$W(x_0) = (1 - \delta^*) \sum_{t=0}^{\infty} (\delta^*)^t V(x_t^*, x_{t+1}^*).$$

For any $\delta \in \mathcal{D}$, it is obtained that:

$$\begin{aligned} W(x_0) &\leq (1 - \delta)V(x_0, x_1^*) + \delta W(x_1^*) \\ &\leq (1 - \delta)V(x_0, x_1^*) + \delta((1 - \delta)V(x_1^*, x_2^*) + \delta W(x_2^*)) \\ &\dots \\ &\leq (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t^*, x_{t+1}^*). \end{aligned}$$

Hence

$$\begin{aligned} W(x_0) &\leq \min_{\delta \in \mathcal{D}} (1 - \delta) \sum_{t=0}^{\infty} \delta^t V(x_t^*, x_{t+1}^*) \\ &\leq J(x_0). \end{aligned}$$

Suppose then that $(\tilde{\delta}, \tilde{\chi})$ satisfies

$$v(\tilde{\delta}, \tilde{\chi}) = \max_{\chi \in \Pi(x_0)} \min_{\delta \in \mathcal{D}} v(\delta, \chi).$$

The value $\chi^* = \{x_t^*\}_{t=0}^\infty$ being an optimal solution of the problem $\mathcal{R}(\delta^*)$, it is obtained that:

$$\begin{aligned} v(x_0) &= v(\tilde{\delta}, \tilde{\chi}) \\ &\leq v(\delta^*, \tilde{\chi}) \\ &\leq v(\delta^*, \chi^*) \\ &= W(x_0). \end{aligned}$$

Hence:

$$\begin{aligned} J(x_0) &= \max_{\chi \in \Pi(x_0)} \min_{\delta \in \mathcal{D}} v(\delta, \chi) \\ &= \min_{\delta \in \mathcal{D}} \max_{\chi \in \Pi(x_0)} v(\delta, \chi) \\ &= W(x_0), \end{aligned}$$

that establishes the details of the statement.

QED

H. PROOF OF PROPOSITION 4

Proof. Denote by φ_δ the optimal policy function of $\mathcal{R}(\delta)$.

Consider the case $x_o < x^\delta$. The system of inequalities $x_o < \varphi_{\underline{\delta}}(x_o) < x^\delta$ is available, hence

$$\begin{aligned} W(x_o) &= J_{\underline{\delta}}(x_o) \\ &= (1 - \underline{\delta})V(x_o, \varphi_{\underline{\delta}}(x_o)) + \underline{\delta}J(\varphi_{\underline{\delta}}(x_o)) \\ &= (1 - \underline{\delta})V(x_o, \varphi_{\underline{\delta}}(x_o)) + \underline{\delta}W(\varphi_{\underline{\delta}}(x_o)) \\ &= \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_o, \varphi_{\underline{\delta}}(x_o)) + \delta W(\varphi_{\underline{\delta}}(x_o))]. \end{aligned}$$

It remains to prove that

$$W(x_o) \geq \min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_o, x_1) + \delta W(x_1)]$$

for any $x_1 \in \Gamma(x_o)$.

But, and for every $x_1 \in \Gamma(x_o)$,

$$\min_{\delta \in \mathcal{D}} [(1 - \delta)V(x_o, x_1) + \delta W(x_1)] \leq (1 - \underline{\delta})V(x_o, x_1) + \underline{\delta}W(x_1).$$

The function $(1 - \underline{\delta})V(x_o, x_1) + \underline{\delta}W(x_1)$ being strictly concave in x_1 , it hence attains its maximum at $x_1 = \varphi_{\underline{\delta}}(x_o)$.

Recall indeed that $x_o < \varphi_{\underline{\delta}}(x_o) < x^\delta$, and $W(\varphi_{\underline{\delta}}(x_o)) = J_{\underline{\delta}}(\varphi_{\underline{\delta}}(x_o))$ and $W'(\varphi_{\underline{\delta}}(x_o)) = J'_{\underline{\delta}}(\varphi_{\underline{\delta}}(x_o))$. This implies

$$(1 - \underline{\delta})V_2(x_o, \varphi_{\underline{\delta}}(x_o)) + \underline{\delta}W'(\varphi_{\underline{\delta}}(x_o)) = 0.$$

From the strict concavity of $(1 - \underline{\delta})V(x_o, x_1) + \underline{\delta}W(x_1)$ and for any $x_1 \in \Gamma(x_o)$:

$$(1 - \underline{\delta})V(x_o, x_1) + \underline{\delta}W(x_1) \leq (1 - \underline{\delta})V(x_o, \varphi_{\underline{\delta}}(x_o)) + \underline{\delta}W(\varphi_{\underline{\delta}}(x_o)).$$

The same arguments can be used for the remaining cases $x^\delta \leq x_o \leq x^{\bar{\delta}}$ and $x^{\bar{\delta}} \leq x_o$ and the argument of the proof is complete. QED

I. PROOF OF THE CONCLUSION IN EXAMPLE 7.2

Proof. The monotonicity of a solution to $\mathcal{R}(\delta)$ will be used in order to prove that $W(x_o) = J(x_o)$. Obviously, the inequality $W(x_o) \leq J(x_o)$ keeps on being satisfied.

The equality will first be proved in the case $x_o < x^{\delta^*}$. Like in the proof of Proposition 3, define $\chi^* = \{x_t^*\}_{t=0}^\infty = \{\varphi^t(x_o)\}_{t=0}^\infty$, the optimal sequence generated from functional equation, and define $\hat{\chi} = \{\hat{x}_t\}_{t=0}^\infty$ the solution of $\mathcal{R}(\underline{\delta})$. Observe that $\{\hat{x}_t\}_{t=0}^\infty$ is strictly increasing.

Suppose that $\chi^* \neq \hat{\chi}$. The same arguments as in the proof of Proposition 3 can be used, the only difference being that λ is to be selected sufficiently closed from 0. Since the sequence $\{W(\hat{x}_t)\}$ is uniformly bounded, T can be fixed as being large enough in order to satisfy:

$$(1 - \underline{\delta}) \sum_{t=0}^T \underline{\delta}^t V(\hat{x}_t, \hat{x}_{t+1}) + \underline{\delta}^{T+1} W(\hat{x}_{T+1}) > (1 - \underline{\delta}) \sum_{t=0}^\infty \underline{\delta}^t V(x_t^*, x_{t+1}^*).$$

For $0 \leq \lambda \leq 1$, define $x_t^\lambda = (1 - \lambda)\hat{x}_t + \lambda x_t^*$. Recall that

$$V(x_t^*, x_{t+1}^*) < W(x_{t+1}^*) \text{ for every } t \geq 0.$$

Take λ sufficiently close to 0 such that for any $0 \leq t \leq T$, $V(x_t^\lambda, x_{t+1}^\lambda) < W(x_{t+1}^\lambda)$ prevails: for any $0 \leq t \leq T$,

$$\operatorname{argmin}_{\delta \in \mathcal{D}} [(1 - \delta)V(x_t^\lambda, x_{t+1}^\lambda) + \delta W(x_{t+1}^\lambda)] = \{\underline{\delta}\}.$$

By applying the same arguments as in the proof of Proposition 3, a contradiction is available. The establishment of equality between W and J is given like in the proof of Theorem 1.

The same arguments can be used for the cases $x^{\delta^*} \leq x_o \leq x^{\bar{\delta}^*}$ or $x_o > x^{\bar{\delta}^*}$. QED

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