Intertemporal equilibrium with heterogeneous agents, endogenous dividends and borrowing constraints*

Stefano BOSI† Cuong LE VAN‡ Ngoc-Sang PHAM§

April 17, 2016

Abstract

We consider a dynamic general equilibrium model with heterogeneous producers and financial market imperfections (borrowing constraints). First, we prove the existence of equilibrium and we give a tractable characterization to check whether a sequence is an equilibrium. Second, we study the effects of financial imperfections on economic growth and land prices. Third, we introduce the notion of (endogenous) land dividends (yields) and new concepts of land bubbles (standard, individual and strong). Some examples of land bubbles are provided in deterministic economies with and without short-sales.

Keywords: Infinite-horizon, general equilibrium, financial market imperfection, land bubbles.

JEL Classifications: C62, D53, D9, E44, G10.

1 Introduction

This paper studies the impact on economic activity of financial market imperfections and land bubbles with endogenous dividends. The general equilibrium theory is pertinent to address this issue and capture the interplay between financial and real markets. Rational bubbles were introduced in overlapping generations general equilibrium models since the early Eighties. Today, a growing attention is paid to infinite-horizon economies. Indeed, the finite-life span of generations introduces an artificial imperfection in OLG models, while infinite-horizon models are more suitable to represent short-run business cycles.

We consider an infinite-horizon general equilibrium model with three assets: a consumption good, land to produce this good, and a financial asset with zero supply. Agents differ in terms of endowments, technology, preferences and borrowing limits. Their number is finite.

In each period, any agent may produce, exchange financial assets and consumption good (endowments and land fruits) and consumes. In the spirit of Geanakoplos and Zame (2002) and Kiyotaki and Moore (1997), agents issue financial assets within the borrowing limits fixed by land as collateral. More precisely, the refund may not exceed a given fraction of land income.[1] This fraction reflects the borrowing limit of agents.

*The authors acknowledge the financial support of the LabEx MME-DII (no. ANR11-LBX-0023-01) and the Institut Europlace de Finance Louis Bachelier.

†EPEE, University of Evry. Email: stefano.bosi@univ-evry.fr

‡IPAG, CNRS, and Paris School of Economics. Email: Cuong.Le-Van@univ-paris1.fr

§Corresponding author. EPEE, University of Evry. Email: pns.pham@gmail.com.

[1] This income is the sum of the value of land and its fruit.

Before studying the equilibrium properties, we show the existence. A proof à la Becker, Bosi, Le Van, and Seegmuller (2015) or Le Van and Pham (2015) no longer applies because agents trade short-lived financial assets with zero supply instead of long-lived assets. The challenge is to prove that individual asset volumes are bounded. To overcome the difficulty, we introduce an intermediate economy where the real asset is replaced by a nominal one. In this economy, we can bound the volume of financial asset and, therefore, adapt the proof of existence à la Becker, Bosi, Le Van, and Seegmuller (2015). We end up by building an equilibrium in the original economy from the one in the intermediate economy. The indirect proof applies to a large class of general equilibrium models. It contributes to the novelty of the paper.

The standard literature (Lucas, 1978; Kocherlakota, 1992; Santos and Woodford, 1997) considers the following long-lived asset : a unit purchased today delivers an exogenous amount of consumption good (a real dividend) tomorrow. In our model, any agent may produce with a landowner-specific technology which may be non-linear. Hence, one unit of land now yields an endogenous amount of consumption good. This amount we call *dividend of land*. Formally, land dividends (d_t) are defined by the following asset-pricing equation:

$$\frac{q_t}{p_t} = \gamma_{t+1}(\frac{q_{t+1}}{p_{t+1}} + d_{t+1}) \quad (1)$$

where p_t and q_t are the prices of consumption good and land and γ_{t+1} is an endogenous discount factor. (1) is a non-arbitrage condition: what we pay today to buy 1 unit of land is equal to what we will receive by reselling 1 unit of land plus land dividends (in terms of consumption good).

In our economy, land plays three different roles: once we buy land, we can (1) resell it, (2) use it to produce, and (3) use it to borrow (collateral role). The land dividend represents the two last roles. At equilibrium, land dividend is between the lowest and the highest marginal productivities. When agents share the same linear technology, we recover the Lucas' tree with exogenous dividends.

The general equilibrium perspective is suitable to represent the interference between financial markets and economic activities. We focus on the financial trade and production of goods by means of land. Any agent may produce giving rise to a competition in the financial markets. Moreover, lenders and borrowers are endogenously identified

Many equilibrium properties follow, some of whom deserve mention.

(1) If the financial system is good enough (in the sense that borrowing limit of any agent is 1 – everyone has a full access to financial markets), the land dividend equals the highest marginal productivity; in addition, if anybody produces, then agents' marginal productivities turn out to be the same and coincide with the dividend of land. Cases where some agents give up the production are quite specific: they experience a very low productivity while the others, a full access to credit markets. However, we show, with examples, that when agents are prevented from borrowing, the most productive agent may not produce.

(2) Agents with marginal productivity exceeding the land dividend borrow until her borrowing constraint becomes binding.

(3) The steady state analysis is carried out in a simple case.[2] In the long run, the most patient agent may not hold the entire stock of land. This result challenges the well-known Ramsey's conjecture (the most patient individual owns all the capital in the long run) and looks more realistic. The very reason is that, in our model, any agent is a producer differently from what happens in the growth literature where consumers rent capital to a representative firm (see Becker and Mitra (2012) among others).

[2]The steady state may not exist when endowments are not stationary.

The last part of the paper focuses on rational land bubbles and their economic consequences. Given an equilibrium, the bubble is the (positive) difference between the asset price and its fundamental value. At first blush, this standard definition looks simple and sound, but it hides the complexity of the notion of fundamental value. Indeed, different notions exist depending on the asset structure and the discount factors we choose to define the fundamental value of the asset. In our model, we remove any ambiguity about the notion of bubble, by providing a clear definition of fundamental value based on a clear definition of dividends and discount factors.

As regards the seminal literature, Tirole (1982), Kocherlakota (1992), Santos and Woodford (1997) consider an asset bringing a sequence of exogenous dividends and define the fundamental value of this asset as the sum (over time) of discounted values of dividends. In our model, the nature of asset is different: any agent can use land to produce the consumption good and the sequence of land dividends is endogenously determined by the asset-pricing equation (1). In the spirit of the standard literature, the fundamental value of land is defined by the sum of discounted values of these endogenous dividends; in this definition, the discount factors are determined by the interest rates of the financial market. As seen above, land dividends capture the twofold nature of land: input to produce and collateral to borrow. The fundamental value of land also reflects the twofold nature of land services. As above, we said that a land bubble arises whenever the price of land exceeds this fundamental value.

The transversality condition plays a very important role in the existence of bubbles. We contribute to the existing literature by proving the necessary of two different transversality conditions at equilibrium. The first one involves the individual discount factors (or, equivalently, the individual expected interest rates) and represents the optimality of agents' behavior. The other transversality condition involves the discount factors of the economy (or, equivalently, the interest rates of the economy) and captures the effects of market imperfections (borrowing constraints). These two conditions become the same in the absence of market imperfections.

A number of results about bubbles follow.

(1) Land bubbles are ruled out if the borrowing constraints of any agent are not binding from some date on. In other words, bubbles arise only if the borrowing constraints of some agent are binding infinitely many times.[3] Indeed, when the borrowing constraints are not binding, the discount factors of any agent coincide with the discount factor of the economy. In this case, the no-bubble condition is equivalent to a no-Ponzi scheme. Since the transversality conditions are satisfied, the no-bubble condition is verified as well.

(2) When the borrowing limit of any agent equals one (the maximum value), endowments are uniformly bounded from above and the TFPs are bounded away from zero, then, there is no land bubble. This result suggests that rational land bubble only appear in economies with financial frictions and/or endowments grow and/or land technologies collapse.

We provide some new examples of bubbles in economies with and without short-sales. In the first one, agents are prevented from borrowing (with zero borrowing limits), their endowments fluctuate and/or technologies are non-stationary.

Agents face a drop in endowments at time t, but are unable to borrow and transfer wealth from period $t+1$ to period t. Thus, they may buy land at date $t-1$ at a higher price, independent on their technologies, in order to transfer their wealth from date $t-1$ to date t. When the agents TFP goes to zero, the fundamental value of land tends to zero as well, below the above level of price. By consequence, a land bubble arises. In the second example, short-sales are allowed but borrowing constraints still work. As in the first example, at any

[3]This is only a necessary condition for the existence of bubble. This issue will be addressed in Sections 5.1.1 and 5.2.

date, at least one agent is forced to save, hence she may accept to buy land at a higher price or to buy a financial asset with low interest rates. Therefore, bubbles may occur.

(3) We introduce the new concept of individual bubble and bridge it and the notion of land bubble. The *individual land dividend* an agent i expects, is determined by an asset-pricing equation:

$$\frac{q_t}{p_t} = \gamma_{i,t+1}\left(\frac{q_{t+1}}{p_{t+1}} + d_{i,t+1}\right)$$

where $\gamma_{i,t+1}$ denotes the individual discount factor of agent i between date t and date $t+1$. The individual land dividend is composed of production and collateral returns, and it also includes the shadow prices of borrowing constraints. The individual fundamental value of land with respect to agent i is a sum of discounted values of the land dividends expected by the agent i.

An individual bubble with respect to agent i (called *i-bubble*) exists if the price of land exceeds its individual fundamental value. We list three results. (1) If an i-bubble exists, a land bubble exists, but the converse is not true. (2) If the individual fundamental value of any agent equals the fundamental value, bubbles and all i-bubbles are ruled out; however when the individual fundamental values are equals but different from the fundamental value, bubbles may arise. (3) The equilibrium price of land equals the maximum individual fundamental value of land. We also provide an example where an i-bubble exists. Again, binding borrowing constraints are just a necessary condition for the existence of individual bubbles.

These results suggest that the kind of discount factors we choose to evaluate the fundamental value of land, matters for bubbles.

Our definition of land bubble contributes to the literature on bubbles of assets delivering endogenous dividends.[4] Among others, two approaches deserve mention.

Miao and Wang (2012, 2015) consider bubbles on the firm's value with endogenous dividends. They split this value in two parts: $V(K) = QK + B$, where K is the initial stock of capital, Q is the marginal Tobin's Q (endogenous) and B is the bubble. They interpret QK as the fundamental value of firm.

Becker, Bosi, Le Van, and Seegmuller (2015), Bosi, Le Van and Pham (2015) introduces the concept of physical capital bubble in one- and two-sector models. They define the fundamental value of physical capital as the sum of discounted values of capital returns (after depreciation). As above, a physical capital bubble exists if the equilibrium price of physical capital exceeds this fundamental value. In our model, land, as input, looks like capital. Becker, Bosi, Le Van, and Seegmuller (2015) consider a representative firm while, in our work, each agent is viewed as an entrepreneur.

The rest of the paper is organized as follows. Section 2 presents the framework and provides some preliminary equilibrium properties. Sections 3 and 4 study the role of financial market imperfections and land bubbles. Section 5 presents examples of bubbles in deterministic economies with and without short-sales. Section 6 concludes. Technical proofs are gathered in Appendices.

2 Framework

We consider an infinite-horizon general equilibrium model without uncertainty. The time is discrete and runs from date 0 to infinity. The number m of agents is finite. I denotes the set of agents.

[4]We refer to Miao (2014) for an introduction to bubbles in infinite-horizon models. A survey on bubbles with asymmetric information, overlapping generations, heterogeneous beliefs can be found in Brunnermeier and Oehmke (2012).

Consumption good. There is a single consumption good. At each period $t = 0, 1, 2, \ldots$, the price of consumption good is denoted by p_t. Each agent i is endowed with $e_{i,t}$ units of consumption good, and chooses to consume $c_{i,t}$ units of this good.

Land. We denote by L and q_t the (exogenous) total supply of land and its price at date t. At this date, the agent i buys $l_{i,t+1}$ units of land. More precisely, (1) she uses this land to produce $F_i(l_{i,t+1})$ units of consumption good and (2) she resells land at a price q_{t+1}. F_i is the production function of agent i.

The financial market opens at the initial date. If agent i buys $a_{i,t}$ units of financial asset at date $t-1$, she will receive $R_t a_{i,t}$ (in terms of money) at date t, where R_t is the gross return. Agents can also borrow in the credit market. However, if they do that, they are required to hold land as collateral. The sense and the role of borrowing constraints will be explained below.

Each household i takes the sequence of prices $(p, q, R) := (p_t, q_t, R_t)_{t=0}^{\infty}$ as given and chooses sequences of consumption, land, and asset volume $(c_i, l_i, a_i) := (c_{i,t}, l_{i,t+1}, a_{i,t+1})_{t=0}^{+\infty}$ in order to maximizes her intertemporal utility

$$P_i(p, q, R): \quad \max_{(c_i, l_i, a_i)} \sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t})$$

subject to, for each t,

$$l_{i,t+1} \geq 0 \tag{2}$$

$$p_t c_{i,t} + q_t l_{i,t+1} + p_t a_{i,t+1} \leq p_t e_{i,t} + q_t l_{i,t} + p_t F_i(l_{i,t}) + R_t a_{i,t} \tag{3}$$

$$R_{t+1} a_{i,t+1} \geq -f_i \big[q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1}) \big] \tag{4}$$

where $l_{i,0} > 0$ is given. We assume that there is no debt before the opening of financial markets, that is $a_{i,0} = 0$.

Borrowing constraint (4) means that agent i can borrow an amount whose repayment does not exceed an exogenous share of land income. The share f_i is set by law below 1 in order to ensure that the agent's land income exceeds her debt. f_i is the borrowing limit of agent i.

Condition (4) can be reinterpreted as a collateral constraint: agents can borrow but they must own land as collateral. By the way, our model is also related to the literature on general equilibrium with collateral constraints (Geanakoplos and Zame, 2002; Kiyotaki and Moore, 1997). However, it is different from Kiyotaki and Moore (1997) where the repayment does not exceed the revenue from reselling land. In other words, Kiyotaki and Moore (1997) require

$$R_{t+1} a_{i,t+1} \geq -q_{t+1} l_{i,t+1}.$$

Our model is also in the spirit of (Liu, Wang, and Zha, 2013) where land-price dynamics are considered.

Remark 1. *Kiyotaki and Moore (1997) consider two types of agents, a farmer and a gatherer, with different time preferences: $\beta < \beta'$. The farmer has a linear production function, while the gatherer has a decreasing return to scale production function. Kiyotaki and Moore (1997) look at the equilibrium properties around the steady state and assume that R_t/p_t is exogenous and equal to $1/\beta'$ and that $f_i = 1$ for every i. Differently from Kiyotaki and Moore (1997), we do not require these assumptions and we will provide global analysis of intertemporal equilibria.*

Remark 2. *Our model differs also from Farhi and Tirole (2012). Indeed, we consider dynamic firms in an infinite-horizon GE model while they focus on firms living for* 3 *periods in an OLG model.*

Remark 3. *When we consider a linear technology ($F_i(x) = ax$ for every i), we recover the Lucas tree. When $F_i = 0$ for every i, land becomes the pure bubble asset as in Tirole (1985).*

The economy, denoted by $\mathcal{E}$, is characterized by a list of fundamentals

$$\mathcal{E} := (u_i, \beta_i, e_i, f_i, l_{i,0}, F_i).$$

Definition 1. *A list* $\Big(\bar{p}_t, \bar{q}_t, \bar{R}_t, (\bar{c}_{i,t}, \bar{l}_{i,t+1}, \bar{a}_{i,t+1})_{i=1}^m\Big)_{t=0}^{+\infty}$ *is an equilibrium of the economy $\mathcal{E}$ if the following conditions are satisfied:*

(i) Price positivity: $\bar{p}_t, \bar{q}_t, \bar{R}_{t+1} > 0$ for $t \geq 0$.

(ii) Market clearing: at each $t \geq 0$,

$$\text{good: } \sum_{i=1}^{m} \bar{c}_{i,t} = \sum_{i=1}^{m} (e_{i,t} + F_i(\bar{l}_{i,t})) \tag{5}$$

$$\text{land: } \sum_{i=1}^{m} \bar{l}_{i,t} = L \tag{6}$$

$$\text{financial asset: } \sum_{i=1}^{m} \bar{a}_{i,t} = 0. \tag{7}$$

(iii) Agents' optimality: for each i, $(\bar{c}_{i,t}, \bar{l}_{i,t+1}, \bar{a}_{i,t+1})_{t=0}^{\infty}$ is a solution of the problem $P_i(\bar{p}, \bar{q}, \bar{R})$.

Note that the financial asset in our framework is a short-lived asset with zero supply, which is different from the long-lived asset bringing exogenous positive dividends in Kocherlakota (1992), Santos and Woodford (1997), Le Van and Pham (2015).

2.1 The existence of equilibrium

In what follows, if we do not explicitly mention, we will work under the following assumptions.

Assumption 1 (production functions)**.** *For each i, the function F_i is concave, continuously differentiable, $F_i' > 0$ and $F_i(0) = 0$.*

Assumption 2 (endowments)**.** *$l_{i,0} > 0$ for any i, and $e_{i,t} > 0$ for any i and for any t.*

Assumption 3 (borrowing limits)**.** *$f_i > 0$ for any i.*

Assumption 4 (utility functions)**.** *For each i, the function u_i is continuously differentiable, concave, $u_i' > 0$, $u_i'(0) = \infty$.*[5]

Assumption 5 (finite utility)**.** *For each i*

$$\sum_{t=0}^{\infty} \beta_i^t u_i(W_t) < \infty, \tag{8}$$

where $W_t := \sum_{i=1}^{m} \Big(e_{i,t} + F_i(L)\Big)$.

[5]In the proof of the existence of equilibrium, we do not require $u_i'(0) = \infty$. This condition is to ensure that $c_{i,t} > 0$ for any t, which is used in next Sections.

Proposition 1. *Under the above assumptions, there exists an equilibrium.*

We cannot directly use a method of Becker, Bosi, Le Van, and Seegmuller (2015) or Le Van and Pham (2015) because the financial asset in our model is a short-lived asset with zero supply. The difficulty is to prove that the asset volume $a_{i,t}$ is bounded. To overcome this difficulty, we introduce an intermediate economy with a nominal asset whose structure is different from that of the financial asset in the original economy. In this intermediate economy, we can bound the volume of the financial asset, and so can prove the existence of equilibrium by adapting the method of Becker, Bosi, Le Van, and Seegmuller (2015) and Le Van and Pham (2015): (1) we prove the existence of equilibrium for each $T-$ truncated economy $\mathcal{E}^T$; (2) we show that this sequence of equilibria converges for the product topology to an equilibrium of our economy $\mathcal{E}$.

Last, we construct an equilibrium for the original economy from an equilibrium of the intermediate economy.

Let us introduce the intermediate economy $\tilde{\mathcal{E}}$ as follows. We only change the structure of the financial asset. We consider a nominal asset b with the sequence of returns $(r_t)_{t\geq 1}$. In this economy, each household i takes the sequence of prices $(p,q,r) = (p_t,q_t,r_t)_{t=0}^{\infty}$ as given and chooses sequences of consumption, land, and asset volume $(c_i,l_i,b_i) := (c_{i,t},l_{i,t+1},b_{i,t+1})_{t=0}^{+\infty}$ in order to maximizes her intertemporal utility $\sum\limits_{t=0}^{+\infty}\beta_i^t u_i(c_{i,t})$ subject to sequences of budget and borrowing constraints. Her maximization problem is

$$\tilde{P}_i(p,q,r): \quad \max \sum_{t=0}^{+\infty}\beta_i^t u_i(c_{i,t}) \tag{9}$$

subject to, for each t,

$$\begin{aligned}
&l_{i,t+1} \geq 0\\
&p_t c_{i,t} + q_t l_{i,t+1} + b_{i,t+1} \leq p_t e_{i,t} + q_t l_{i,t} + p_t F_i(l_{i,t}) + r_t b_{i,t}\\
&r_{t+1} b_{i,t+1} \geq -f_i\big[q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1})\big]
\end{aligned} \tag{10}$$

where $l_{i,0} \geq 0$ is given and there is no debt before the opening of the financial market, that is $b_{i,0} = 0$.

Definition 2. *A list* $\Big(\bar{p}_t,\bar{q}_t,\bar{r}_t,(\bar{c}_{i,t},\bar{l}_{i,t+1},\bar{b}_{i,t+1})_{i=1}^m\Big)_{t=0}^{+\infty}$ *is an equilibrium of the economy* $\tilde{\mathcal{E}}$ *if the following conditions are satisfied:*

(i) $\bar{p}_t,\bar{q}_t,\bar{r}_{t+1} \in (0,+\infty)$ *for any* $t \geq 0$.

(ii) Market clearing: at each $t \geq 0$, *we have* $\sum_{i=1}^m \bar{c}_{i,t} = \sum_{i=1}^m (e_{i,t} + F_i(\bar{l}_{i,t}))$, $\sum_{i=1}^m \bar{l}_{i,t} = L$, *and* $\sum_{i=1}^m \bar{b}_{i,t} = 0$.

(iii) Agents' optimality: for each i, $(\bar{c}_{i,t},\bar{l}_{i,t+1},\bar{b}_{i,t+1})_{t=0}^{\infty}$ *is a solution of the problem* $\tilde{P}_i(\bar{p},\bar{q},\bar{r})$.

By changing variables ($a_{i,t} := b_{i,t}/p_{t-1}, R_t := r_t p_{t-1}$), it is easy to prove the following result.

Lemma 1. *If* $\Big(\bar{p}_t,\bar{q}_t,\bar{r}_t,(\bar{c}_{i,t},\bar{l}_{i,t+1},\bar{b}_{i,t+1})_{i=1}^m\Big)_{t=0}^{+\infty}$ *is an equilibrium for the economy* $\tilde{\mathcal{E}}$ *then* $\Big(\bar{p}_t,\bar{q}_t,\bar{R}_t,(\bar{c}_{i,t},\bar{l}_{i,t+1},\bar{a}_{i,t+1})_{i=1}^m\Big)_{t=0}^{+\infty}$, *where* $\bar{a}_{i,t} := \bar{b}_{i,t}/\bar{p}_{t-1}, \bar{R}_t := \bar{r}_t\bar{p}_{t-1}$, *is an equilibrium for the economy* $\mathcal{E}$.

Remark 4. *The crucial point is the following: if we choose the set of prices is the set of positive vectors such that $2p_t + q_t + r_t = 1 \forall t$, then, since $l_{i,t} \leq L$ for any i, t, we can use (10) and the induction argument, to prove that: the asset volume $b_{i,t}$ is bounded from above by an exogenous bound. We cannot do so in the original economy $\mathcal{E}$. See Appendix 11 for a proof of the existence of equilibrium for the economy $\tilde{\mathcal{E}}$.*

Note that in this proof, we allow for non-stationary production functions. However, in this paper (except Section 5.1.1), we assume that the technology is stationary for the sake of simplicity.

2.2 Borrowing constraints and transversality conditions

We provide the following fundamental result: a tractable necessary and sufficient condition to verify that a sequence is an intertemporal equilibrium.

Proposition 2. *1. Let $\Big(p, q, R, (c_i, l_i, a_i)_{i=1}^m\Big)$ be an equilibrium. There exists a positive sequence of multipliers $(\lambda_{i,t}, \eta_{i,t+1}, \mu_{i,t+1})$ such that*

$$\text{FOCs: } \beta_i^t u_i'(c_{i,t}) = \lambda_{i,t} p_t \tag{11}$$

$$\lambda_{i,t} p_t = (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} \tag{12}$$

$$\lambda_{i,t} q_t = (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})) + \eta_{i,t+1} \tag{13}$$

$$\eta_{i,t+1} l_{i,t+1} = 0 \tag{14}$$

$$\mu_{i,t+1}\Big(R_{t+1} a_{i,t+1} + f_i \big[q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1})\big]\Big) = 0 \tag{15}$$

$$\text{Transversality condition: } \lim_{t \to \infty} \lambda_{i,t} p_t \Big(\frac{q_t}{p_t} l_{i,t+1} + a_{i,t+1}\Big) = 0 \tag{16}$$

Moreover, we have, for any i,

$$\infty > \sum_{t=0}^{\infty} \lambda_{i,t} p_t c_{i,t} = \lambda_{i,0} p_0 \big(F_i(l_{i,0}) + \frac{q_0}{p_0} l_{i,0}\big) + \sum_{t=0}^{\infty} \lambda_{i,t} p_t e_{i,t}$$
$$+ \sum_{t=1}^{\infty} \lambda_{i,t} p_t \Big(1 + f_i \frac{\mu_{i,t}}{\lambda_{i,t}}\Big)\Big(F_i(l_{i,t}) - l_{i,t} F_i'(l_{i,t})\Big) \tag{17}$$

2. If the sequences (c_i, a_i, l_i, p, q, R) and $(\lambda_i, \eta_i, \mu_i)$ satisfy

(a) $c_{i,t}, l_{i,t}, \lambda_{i,t}, \eta_{i,t+1}, \mu_{i,t+1} \geq 0$; $p_t, q_t, R_t > 0$;

(b) conditions (3) is binding, and (2), (4), (5), (6), (7) hold;

(c) conditions (11), (12), (13), (14), (15), (16) hold;

(d) $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) < \infty$;

then (c_i, a_i, l_i, p, q, R) is an intertemporal equilibrium.

The challenge and key point of Proposition 2 is the necessity of transversality conditions (16). To prove this, we develop the method in Kamihigashi (2002). Recall that Kamihigashi (2002) only considers positive allocations while asset volume $a_{i,t}$ may be negative in our model. The detailed proof of this result is presented in Appendix 7.

Remark 5. *1. In the light of transversality condition (16), we keep the price p_t to avoid any confusion. When everything is clear, we will normalize by setting $p_t = 1$ for any t.*

2. Since we allow for $F_i'(0) < \infty$, there may be some agent who doesn't use land to produce.

According to (12), we have $\lambda_{i,t}p_t \geq \lambda_{i,t+1}R_{t+1}$ for every i. Since $f_i > 0$ for any i, it is easy to see that there exists an agent i whose borrowing constraint (4) is not binding. Thus $\mu_{i,t+1} = 0$ which implies that $\lambda_{i,t}p_t = \lambda_{i,t+1}R_{t+1}$. As a result, we get the following result:

Lemma 2. *We have*

$$\frac{p_{t+1}}{R_{t+1}} = \max_{i\in\{1,\dots,m\}} \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} \tag{18}$$

We define the discount factor γ_{t+1} ($\gamma_{i,t+1}$) of the economy (agent i) from date t to date $t+1$, and the discount factor Q_t ($Q_{i,t}$) of the economy (agent i) from the initial date to date t as follows

$$\gamma_{t+1} := \max_{i\in\{1,\dots,m\}} \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}, \quad Q_0 := 1, \quad Q_t := \gamma_1 \dots \gamma_t$$
$$\gamma_{i,t+1} := \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}, \quad Q_{i,0} := 1, \quad Q_{i,t} := \gamma_{i,1} \cdots \gamma_{i,t} = \frac{\beta_i^t u_i'(c_{i,t})}{u_i'(c_{i,0})}$$

Note that $\gamma_{i,t} \leq \gamma_t$ for any i and t.

We rewrite borrowing constraint (4) as

$$Q_{t+1}\frac{R_{t+1}}{p_{t+1}}a_{i,t+1} \geq -f_i Q_{t+1}\big[\frac{q_{t+1}}{p_{t+1}}l_{i,t+1} + F_i(l_{i,t+1})\big]$$

According to definition of (Q_t) and Lemma 2, we see that $Q_t = \dfrac{R_{t+1}}{p_{t+1}}Q_{t+1}$. Therefore, borrowing constraint (4) is equivalent to

$$Q_t a_{i,t+1} \geq -f_i Q_{t+1}\big[\frac{q_{t+1}}{p_{t+1}}l_{i,t+1} + F_i(l_{i,t+1})\big]$$

Corollary 1 (fluctuation of borrowing constraints)**.** *At equilibrium, we have:*

1. For each i, there are only two cases:

(a) $\lim_{t\to\infty}\Big(Q_t a_{i,t+1} + f_i Q_{t+1}\big(\frac{q_{t+1}}{p_{t+1}}l_{i,t+1} + F_i(l_{i,t+1})\big)\Big)$ does not exist;

(b) $\lim_{t\to\infty}\Big(Q_t a_{i,t+1} + f_i Q_{t+1}\big(\frac{q_{t+1}}{p_{t+1}}l_{i,t+1} + F_i(l_{i,t+1})\big)\Big) = 0$.

2. (transversality condition, version 2.) We have, for each i,

$$\liminf_{t\to\infty}\Big(Q_t a_{i,t+1} + f_i Q_{t+1}\big(\frac{q_{t+1}}{p_{t+1}}l_{i,t+1} + F_i(l_{i,t+1})\big)\Big) = 0 \tag{19}$$

Remark 6. *We observe that there are two kinds of transversality conditions. The first one is (16) which is determined by the individual discount factor $\beta_i^t u_i'(c_{i,t})/u_i'(c_{i,0})$. It characterizes the optimality of agent i's allocations. The second one is (19) based on the economy discount factor Q_t. It clarifies the role of borrowing constraints.*

Remark 7. *All the results in this section apply also to non-stationary production functions.*

3 The role of the financial market

For each $t \geq 0$, we introduce two productive bounds:

$$\underline{d}_t := \min_{i \in \{1,\dots,m\}} F_i'(l_{i,t}), \quad \bar{d}_t := \max_{i \in \{1,\dots,m\}} F_i'(l_{i,t}).$$

We have the following result showing the relationship among land prices, marginal productivities, interest rates and borrowing limits.

Lemma 3. *The relative price of land is governed by the following inequalities:*

$$\gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + \underline{d}_{t+1}\big) \leq \frac{q_t}{p_t} \leq \gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + \bar{d}_{t+1}\big) \tag{20}$$

$$f_i\gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\big) \leq \frac{q_t}{p_t} \tag{21}$$

for any i and t.

According to (20), we introduce the land dividends.

Definition 3 (**dividends of land**)**.** *The dividends of land $(d_t)_t$ is defined by the following non-arbitrage condition*

$$\frac{q_t}{p_t} = \gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + d_{t+1}\big) \tag{22}$$

Interpretation. Once we buy land, we will be able to resell land and expect to receive an amount. This amount is exactly the dividend of land defined by (22). Equation (22) is a non-arbitrage condition: what we pay to buy 1 unit of land at date t is equal to what we receive by reselling 1 unit of land plus the dividend of land (in terms of consumption good). When technologies are linear and identical ($F_i(X) = dX$ for any i), we have $d_t = d$ for any t, and hence we recover the Lucas' tree. In our general setup, (20) implies that any land dividend (d_t) is greater than the lowest marginal productivity ($\underline{d}_t$) but less than the highest one ($\bar{d}_t$).

In our model, land has a threefold structure: after buying land at date t, agents (1) resell it at date $t+1$, (2) use it as collateral in order to borrow from financial markets and (3) receive an amount of consumption good from their production process. Definition 3 states that dividends are endogenous and capture the roles (2) and (3) of land. In fact, because land is resold and gives dividends at each date, (22) can be interpreted as an asset-pricing or a no-arbitrage condition.

We point out some interesting properties of land dividends.

Lemma 4 (fair financial system)**.** *$d_{t+1} = \bar{d}_{t+1}$ if $f_i = 1$ for any i or (4) is not binding for any i.*

We can interpret $f_i = 1$ as a full access to credit market for agent i. Lemma 4 points out that the land dividend equals the highest marginal productivity if either anyone may fully enter the credit market or borrowing constraints of any agent are not binding.

This following result shows that dividends equal the lowest marginal productivities if every agent buys land.

Proposition 3. *Focus on date t and assume $l_{i,t} > 0$ for every i. In this case, $d_t = \underline{d}_t$.*

We highlight some consequences of Lemma 4 and Proposition 3.

Corollary 2. *1. If $F_i'(0) = +\infty$ for every i, then $d_t = \underline{d}_t$ $\forall t$.*

2. (equal marginal productivities.) If $\mu_{i,t} = f_i \mu_{i,t}$[6] *and $l_{i,t} > 0$, then $F_i'(l_{i,t}) = d_t$ $\forall i$.*

3. If $f_i = 1$ and $F_i'(0) = \infty$ for any i, then $d_t = F_i'(l_{i,t})$ $\forall i, t$.

3.1 Who buys land? Who needs credits?

In this section, we identify agents either as producers or borrowers. Land demand depends on its productivity.

Proposition 4. *If $l_{i,t+1} > 0$ then $F_i'(l_{i,t+1}) \geq d_{t+1}$.*
If $F_i'(l_{i,t+1}) > d_{t+1}$, then borrowing constraint (4) of agent i is binding.

The first statement means that if an agent buys land, its marginal productivity must be greater than land dividends.

The second one shows that if an agent has a marginal productivity which is strictly greater than land dividend, she will borrow until her borrowing constraints become binding. In other words, this agent needs credit.

The following result suggests that agents with a low productivity do not buy land to produce.

Proposition 5. *Focus on agent i and assume that there exists an agent j such that $f_j = 1$ and $F_i'(0) < F_j'(L)$. We have $l_{i,t} = 0$ for every t.*

Note that Proposition 5 holds whatever the form of utility functions and the size of the discount rate β_i.

We can interpret $f_j = 1$ as a full access of agent j to credit market. In this case, any agent i with lower productivity ($F_i'(0) < F_j'(L)$) never produces. Proposition 5 is in line with Proposition 1 in Le Van and Pham (2015) where they prove that nobody invests in the productive sector if the productivity of this sector is too low.

Agents can be reinterpreted as countries. In this case, our economy works as a world economy with free trade. Each country i is endowed with $l_{i,0}$ units of land. When the trade is fully free and the international financial market is good enough (in the sense that $f_i = 1$ for any i), countries with a lower productivity never produce and land in these countries will be held by the countries with the highest productivity.

Remark 8. *When there exist financial (or political) frictions characterized by $f_j < 1$, the analysis becomes more complex. In Section 5.1.1, we will present an example where there are two agents: A and B with $f_A = 0$ (agent A is prevented from borrowing) with linear technologies. In this example, at date $2t+1$, the productivity of agent A is higher than that of agent B, but agent B may produce at date $2t+1$.*

3.2 A particular case: a steady state analysis

In this section, we assume that agents have no endowments, that is $e_{i,t} = 0$ for every i and t. For simplicity, we also assume that there are two agents, say i and j, with different rates of time preference: $\beta_i < \beta_j$.

We give an analysis at the steady state. Recall that when endowments are not stationary, the existence of steady state may not hold.

[6] This condition is satisfied if $f^i = 1$ or $R_t a_{i,t} > -f_i\big[q_t l_{i,t} + p_t F_i(l_{i,t})\big]$, i.e, the borrowing constraint of agent i at date t is not binding.

Lemma 5. *Consider two agents i and j with $\beta_i < \beta_j$. If $e_{i,t} = 0$ for any i and t, and $F_i(l_i) = A_i l_i^{\alpha}$, where $\alpha \in (0,1)$, for any i, then there is a unique steady state (up to a scalar for prices):*

$$\frac{R}{p} = \frac{1}{\beta_j} \tag{23}$$

$$\big(\frac{q}{p}\big)^{\frac{1}{1-\alpha}} L = \Big(\frac{\alpha A_i}{\frac{1}{\beta_i + f_i(\beta_j - \beta_i)} - 1}\Big)^{\frac{1}{1-\alpha}} + \Big(\frac{\alpha A_j}{\frac{1}{\beta_j} - 1}\Big)^{\frac{1}{1-\alpha}} \tag{24}$$

$$l_i = \Big(\frac{\alpha A_i}{\frac{1}{\beta_i + f_i(\beta_j - \beta_i)} - 1}\frac{p}{q}\Big)^{\frac{1}{1-\alpha}}, \quad l_j = L - l_i \tag{25}$$

$$Ra_i + f_i\big[ql_i + pF_i(l_i)\big] = 0, \quad a_i + a_j = 0. \tag{26}$$

Who will own land in the long run?

Cobb-Douglas technologies imply $l_i, l_j > 0$. Each agent holds a strictly positive amount of land to produce themselves. In this respect, our model differs from Becker and Mitra (2012) where the most patient agent holds the entire stock of capital in the long run. The difference rests on two reasons.

First, in Becker and Mitra (2012), the firm is unique and consumers do not produce. In our model, any agent produces with her own technology and can be viewed as a credit-constrained entrepreneur.

Second, in Becker and Mitra (2012), returns on capital are determined by the marginal productivity of their representative firm. In our framework, land dividends are interpreted as land returns and determined by no-arbitrage condition (22).

Corollary 3 (the role of borrowing limit)**.** *Under conditions in Lemma 5, we have:*

1. *The relative price of land q/p increases in f_i.*
2. *The long-run quantity of fruits, i.e., $Y := F_i(l_i) + F_j(l_j)$ is increasing in the borrowing limit f_i.*

The intuition of point 1 is that when f_i increases, agent i can borrow more and, then, land demand increases in turn raising the price of land at the end.

The point 2 is also intuitive: the higher the level of f_i, the more the quantity the agent with the highest productivity can borrow, and, finally, the more the output produced.

4 Land bubbles

Combining $Q_{t+1} = \gamma_{t+1} Q_t$ with (22), we get

$$Q_t \frac{q_t}{p_t} = Q_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + d_{t+1}\big) \tag{27}$$

and, so,

$$\begin{aligned} \frac{q_0}{p_0} &= \gamma_1\big(\frac{q_1}{p_1} + d_1\big) = Q_1 d_1 + Q_1 \frac{q_1}{p_1} \\ &= Q_1 d_1 + Q_1 \gamma_2 \big(\frac{q_2}{p_2} + d_2\big) = Q_1 d_1 + Q_2 d_2 + Q_2 \frac{q_2}{p_2} \\ &= \ldots = \sum_{t=1}^{T} Q_t d_t + Q_T \frac{q_T}{p_T} \end{aligned} \tag{28}$$

for any $T \geq 1$. This leads the following definition.

Definition 4. *The fundamental value of the land is defined by*

$$FV_0 := \sum_{t=1}^{\infty} Q_t d_t$$

Remark 9. *As seen above, land dividends capture a twofold role of land: land is used to produce a consumption good and, as collateral, to borrow. The fundamental value of land reflects the value of both these roles.*

Definition 5 (bubble)**.** *Land bubbles exists if the market price of land (in term of consumption good) exceeds its fundamental value: $q_0/p_0 > FV_0$.*

As in Montrucchio (2004), Le Van and Pham (2014), some equivalences hold.

Proposition 6 (Necessary and sufficient conditions for bubbles)**.** *The following statements are equivalent.*

(i) Land bubbles exist.

(ii) $\lim_{t\to\infty} Q_t q_t/p_t > 0$.

(iii) $\sum_{t=1}^{\infty} (p_t d_t/q_t) < +\infty$.

Note that this result only depends on the non-arbitrage condition (27). It holds for any form of technologies, even non-stationary.

Since we are assuming that technologies are stationary, we have $d_t \geq \min_i F_i'(l_{i,t}) \geq \min_i F_i'(L) > 0$ for every t. This leads the following result.

Corollary 4. *If land bubbles exist, then $\sum_{t=1}^{\infty} (p_t/q_t) < +\infty$.*

This explains why the existence of land bubbles implies that real land prices tend to infinity. Notice, however, that this fact only holds in the case of stationary technology. In Section 5.1.1, this issue will be readdressed.

Interest rates and bubbles. According to (28), we have $\sum_{t=1}^{\infty} Q_t d_t \leq q_0/p_0 < \infty$ and, also, $d_t \geq \min_i F_i'(l_{i,t}) \geq \min_i F_i'(L) > 0$ for every t. Eventually, we get $\sum_{t=1}^{\infty} Q_t < \infty$.

We introduce the real interest rate of the economy ρ_t at date t as follows: $\gamma_t = p_t/R_t = 1/(1+\rho_t)$. We notice that ρ_t may be negative. The condition $\sum_{t=1}^{\infty} Q_t < \infty$ writes explicitly

$$\sum_{t=0}^{\infty} \frac{1}{\prod_{s=1}^{t}(1+\rho_s)} < \infty$$

and we can reinterpret it by saying that the real interest rates are not "too low". We also observe that there exists a sequence of dates $(t_n)_n$ such that $\rho_{t_n} > 0$ for any n.

According to Proposition 6, land bubbles exist if and only if

$$\lim_{t\to\infty} \frac{1}{\prod_{s=1}^{t}(1+\rho_s)} \frac{q_t}{p_t} > 0$$

This condition implies in turn the existence of the limit $\lim_{t\to\infty} \frac{\frac{q_{t+1}}{p_{t+1}}}{\frac{q_t}{p_t}} \frac{1}{1+\rho_{t+1}} = 1.$

Hence, in the long run, if land bubbles exist, the rate of growth of land prices is equal to the gross interest rate.

4.1 No-bubble results

Proposition 7. *If $Q_t/Q_{i,t}$ is uniformly bounded from above for any i, then there are no bubbles.*

Write $\gamma_{i,t} = 1/(1+\rho_{i,t})$, where $\rho_{i,t}$ is interpreted as the the real expected interest rate of agent i at date t. As above, this interest rate may be negative. According to Proposition 7, if a bubble exists, there is an agent i such that her expected interest rates are high with respect to those of the economy in the following sense:

$$\prod_{s=1}^{T} \frac{1+\rho_{i,t}}{1+\rho_t} \xrightarrow{T\to\infty} \infty$$

Let us point out some consequences of Proposition 7.

Corollary 5. *If there exists $T > 0$ such that $\mu_{i,t} = 0$ for any i and $t \geq T$, then there is no land bubble.*

The intuition of this result is that when $\mu_{i,t} = 0$ for any i and $t \geq T$, the individual discount factors coincide with the discount factors of the economy. In this case, the no-bubble condition turns out to be equivalent to the no-Ponzi scheme. Since the transversality conditions are satisfied, the no-bubble condition holds as well.

Corollary 5 implies that if the borrowing constraints of any agent are not binding, then, there is no bubble. The following corollary clarifies it in other words.

Corollary 6 (bubble existence and borrowing constraints)**.** *If land bubbles exist, there exist an agent i and an infinite sequence of dates $(t_n)_n$ such that the borrowing constraints of agent i are binding at each date t_n, that is, for any t_n,*

$$R_{t_n} a_{i,t_n} = -f_i\big[q_{t_n} l_{i,t_n} + p_{t_n} F_i(l_{i,t_n})\big]$$

Remark 10. *The binding of borrowing constraints is only a necessary condition for the existence of bubble. Sections 5.1.1 and 5.2 provide some examples where borrowing constraints of any agent are frequently binding but bubbles may not exist.*

Remark 11. *The relationship between the existence of bubble and borrowing constraints are questioned in Kocherlakota (1992). He considers the borrowing constraints: $x_{i,t} \geq x$, where $x_{i,t}$ is the asset quantity held by agent i at date t and $x \leq 0$ is an exogenous bound. He claims that* $\liminf_{t\to\infty}(x_{i,t} - x) = 0$ *and interprets it concluding that borrowing constraints of agent i are frequently binding. He did not proved that $x_{i,t} - x = 0$ frequently.*

We define the aggregate output of the economy at date t as $Y_t := \sum_{i=1}^{m} [e_{i,t} + F_i(l_{i,t})]$ and the present value of the aggregate output as $\sum_{t=0}^{\infty} Q_t Y_t$.

The main result of the section rests on the following list of four lemmas whose proofs are gathered in Appendix 9.

Lemma 6. *If* $\sup_{i,t} e_{i,t} < \infty$ *and technologies are stationary, the present value of the aggregate output is finite.*

Lemma 7. *Assume that* $\sup_{i,t} e_{i,t} < \infty$ *and technologies be stationary. Given an equilibrium, we obtain that* $Q_t\big(l_{i,t+1}q_t/p_t + a_{i,t+1}\big)$ *is uniformly bounded from below and from above as well.*

Lemma 8. *Let* $\sup_{i,t} e_{i,t} < \infty$ *and technologies be stationary. Given an equilibrium, the following limits exist:*

$$\lim_{t\to\infty} Q_t\big(\frac{q_t}{p_t} l_{i,t+1} + a_{i,t+1}\big) = \lim_{t\to\infty} \big(Q_t \frac{q_t}{p_t} l_{i,t} + Q_{t-1} a_{i,t}\big) \tag{29}$$

for any i.

Lemma 9. *Let* $\sup_{i,t} e_{i,t} < \infty$ *and technologies be stationary. Given an equilibrium, if there exists* T *such that*

$$f_i\big(\frac{q_t}{p_t} + \frac{F_i(l_{i,t})}{l_{i,t}}\big) l_{i,t} \geq \big(\frac{q_t}{p_t} + d_t\big) l_{i,t} \tag{30}$$

for every $t \geq T$, *then*

$$\lim_{t\to\infty} Q_t\big(a_{i,t+1} + \frac{q_t}{p_t} l_{i,t+1}\big) \leq 0$$

Let us now introduce the main result.

Proposition 8. *Let* $\sup_{i,t} e_{i,t} < \infty$, *technologies be stationary and* $f_i = 1$ *for every* i. *Then, land bubbles are ruled out at equilibrium.*

Remark 12. *We also notice that our result still holds for any technology in the form* $A_{i,t}F_i$ *where* $A_{i,t}$ *is bounded away from zero for any* i.

This proposition points out that there is no land bubble at equilibrium when the financial system is good enough (in the sense that $f_i = 1$ for any i), exogenous endowments are bounded from above and the technology is stationary.

Proposition 8 is in line with the results in Kocherlakota (1992), Santos and Woodford (1997), Huang and Werner (2000) and Le Van and Pham (2014), where they prove that bubbles are ruled out if the present value of aggregate endowments is finite. Indeed, the asset in Kocherlakota (1992) is a particular case of the land in our model when $F_{i,t}(X) = \xi_t X$ for any X. Proposition 8 also shows that land bubbles are ruled out in Kiyotaki and Moore (1997).

Proposition 8 suggests that land bubbles only appear when land TFP of land technologies tends to zero and/or endowments grow without bound and/or agents cannot easily enter the financial market ($f_i < 1$). We will present some examples of bubbles in Section 5, where these conditions are violated.

4.2 New concepts: individual and strong bubbles

According to (13), we find

$$\frac{q_t}{p_t} = \frac{\lambda_{i,t+1}p_{i,t+1}}{\lambda_{i,t}p_t}\big(\frac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\big) + \frac{f_i\mu_{i,t+1}p_{t+1}}{\lambda_{i,t}p_t}\big(\frac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\big) + \frac{\eta_{i,t+1}}{\lambda_{i,t}p_t}$$

We then use (11) and (12) to obtain

$$\frac{q_t}{p_t} = \gamma_{i,t+1}\big(\frac{q_{t+1}}{p_{t+1}} + d_{i,t+1}\big) \tag{31}$$

where

$$d_{i,t+1} := \Big(F_i'(l_{i,t+1}) + \frac{\eta_{i,t+1}}{\lambda_{i,t+1}p_{t+1}}\Big) + f_i\big(\frac{\gamma_{t+1}}{\gamma_{i,t+1}} - 1\big)\big(\frac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\big)$$

We call $d_{i,t+1}$ *the individual dividend of agent i at date $t+1$.* Here $d_{i,t+1}$ includes two terms. The first one is $X_{i,t+1} := F_i'(l_{i,t+1}) + \frac{\eta_{i,t+1}}{\lambda_{i,t+1}p_{t+1}}$ which represents the return from the production process.[7] The second term is $f_i\big(\frac{\gamma_{t+1}}{\gamma_{i,t+1}} - 1\big)\big(\frac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\big)$ which can be interpreted as a collateral return. Note that the collateral return is equal to zero if $f_i = 0$ or the discount factors of agent i and of the economy are identical.

The asset-pricing equation (31) shows the way agent i evaluates the price of land. With the individual discount factor $\gamma_{i,t+1}$, once agent i buys land, she will be able to resell land at a price q_{t+1} and she will expect to receive $d_{i,t+1}$ units of consumption good as dividends. Since the individual discount factor $\gamma_{i,t+1}$ is less than that of economy γ_{t+1}, the individual dividend $d_{i,t+1}$ expected by agent i exceeds the dividend d_{t+1} of the economy.

Using (31) and adopting the same argument in (28), we find that, for each $T \geq 1$,

$$\frac{q_0}{p_0} = \sum_{t=1}^{T} Q_{i,t}d_{i,t} + Q_{i,T}\frac{q_T}{p_T}$$

Definition 6 (individual bubble). *1. $FV_i := \sum_{t=1}^{\infty} Q_{i,t}d_{i,t}$ is the i-fundamental value of land. We say that a i-land bubble exists if $q_0/p_0 > \sum_{t=1}^{\infty} Q_{i,t}d_{i,t}$.*

2. A strong bubble exists if the asset price exceeds any individual value of land, that is $q_0/p_0 > \max_i FV_i$ for any i.

The concept of $i-$bubble is closely related to bubbles of durable goods and collateralized assets in Araujo, Pascoa, and Torres-Martinez (2011). Given an equilibrium, Araujo, Pascoa, and Torres-Martinez (2011) provide asset-pricing conditions (Corollary 1, page 263) based on the existence of what they call *deflators* and *non-pecuniary returns* which are not necessarily unique. Then, they define bubbles associated to each deflators and non-pecuniary returns. In our framework, for each equilibrium, we give closed formulas for two types of deflators (wecall γ_t and $\gamma_{i,t}$ discount factor and individual discount factor respectively). Different from Araujo, Pascoa, and Torres-Martinez (2011), the technology in our paper may be non-linear and non-stationary.

By applying the same argument in Proposition 6, we obtain some equivalences.

Proposition 9. *The following statements are equivalent.*

(i) *i-land bubbles exist.*

(ii) $\lim_{t\to\infty} Q_{i,t}q_t/p_t > 0$.

(iii) $\sum_{t=1}^{\infty} (p_t d_{i,t}/q_t) < +\infty$.

Another added-value of our paper (comparing with Araujo, Pascoa, and Torres-Martinez (2011), Pascoa, Petrassi and Torres-Martinez (2011)) is to study the connection between the concepts of bubble and i-bubble. This is showed in the following result.

[7] Note that $X_{i,t+1}l_{i,t+1} = F_i'(l_{i,t+1})l_{i,t+1}$.

Proposition 10. *1. Always $FV_0 \leq FV_i \leq q_0/p_0$ for any i. By consequence, if an i-land bubble exists for some agent i, then a land bubble exists.*

2. There is an agent i such that her i-bubble is ruled out. Consequently, strong land bubbles are ruled out, that is $q_0/p_0 = \max_i FV_i$.

3. If $FV_0 = FV_i$ for any i, then $FV_0 = FV_i = q_0/p_0$ for any i, that is, there is no room for bubbles nor i-bubble.

Comments and discussions. $FV_0 \leq FV_i \leq q_0/p_0$ follows from the definitions of bubble and i-bubble. The intuition is that, since any agent expects a higher interest rate than that of the economy, the individual value of land expected by any agent will exceed the fundamental value of land. Nevertheless, the converse of point 1 is not true. In Section 5.1.1, we present an example where i-bubble does not exist for any i while bubble may arise.

Points 2 shows that there is an agent whose expected value of land equals its equilibrium price. Point 3 is more intuitive and complements point 2: when any individual value of land coincides with that of economy, both bubble and individual bubbles are ruled out. However, when any individual value of land is identical but different from the fundamental value of land, we do not know whether land bubbles are ruled out.[8]

Our concept of strong bubble is related to the notion of *speculative bubble* in Werner (2014). He considers an asset bringing exogenous dividends in a model with ambiguity. Werner (2014) defines the asset fundamental value under the beliefs of agent i as the sum of discounted expected future dividends under her beliefs. He then says that speculative bubble exists if the asset price is strictly higher than any agent's fundamental value. The readers may ask why strong bubbles are ruled out while speculative bubbles in Werner (2014) may exist. It is hard to compare these two results since the two concepts of bubbles are defined in two different settings (with and without ambiguity).

5 Examples of bubbles

In this section, we contribute to the literature of bubbles by providing some examples where bubbles appear in deterministic economies with and without short-sales.[9] Here, dividends are endogenous determined and may be strictly positive.

5.1 Land bubbles without financial market

Focus on the case where there is no financial market. In this section, non-stationary production functions are considered. Let us rewrite the agent's program. The household i takes the sequence of prices $(p, q) = (p_t, q_t)_{t=0}^{\infty}$ as given and chooses sequences of consumption and land $(c_i, l_i) := (c_{i,t}, l_{i,t+1})_{t=0}^{+\infty}$ in order to maximize her intertemporal utility

$$P_i(p, q) : \quad \max \sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t}) \tag{32}$$

$$\text{subject to, for each } t, : \quad l_{i,t+1} \geq 0 \tag{33}$$

$$p_t c_{i,t} + q_t l_{i,t+1} \leq p_t e_{i,t} + q_t l_{i,t} + p_t F_{i,t}(l_{i,t}), \tag{34}$$

[8]See observation "1. bubble vs i-bubble" in Section 5.1.1.

[9]Araujo, Novinski, and Pascoa (2011) provide some examples of equilibria with bubbles in models where the utility functions take the form $\sum_{t\geq 0} \zeta_{i,t} u(c_{i,t}) + \epsilon_i \inf_{t\geq 0} u_i(c_{i,t})$. The parameter ϵ_i plays the key role.

where $l_{i,0} > 0$ is given.

Under a linear technology ($F_{i,t}(x) = \xi_t x$ for every i), the land structure becomes the same asset structure as in Kocherlakota (1992), Santos and Woodford (1997) and Huang and Werner (2000). If $F_i = 0$ for every i, land becomes a pure bubble as in Tirole (1985).

Definition 7. *A list* $\Big(\bar{p}_t, \bar{q}_t, (\bar{c}_{i,t}, \bar{l}_{i,t+1})_{i=1}^m\Big)_{t=0}^{+\infty}$ *is an equilibrium of the economy without financial market under the following conditions.*

(i) Price positivity: $\bar{p}_t, \bar{q}_t > 0$ *for* $t \geq 0$.

(ii) Market clearing: at each $t \geq 0$,

$$\sum_{i=1}^m \bar{c}_{i,t} = \sum_{i=1}^m (e_{i,t} + F_{i,t}(\bar{l}_{i,t})), \quad \sum_{i=1}^m \bar{l}_{i,t} = L.$$

(iii) Agents' optimality: for each i, $(\bar{c}_{i,t}, \bar{l}_{i,t+1})_{t=0}^{\infty}$ *is a solution of the problem* $P_i(\bar{p}, \bar{q})$.

Remark 13. *By applying Proposition 2 and Lemma 10 (Appendix 10), we can check that an equilibrium for the economy without financial market is a part of an equilibrium for the economy* $\mathcal{E}$ *with* $f_i = 0$ *for any* i.

Let $\Big(p, q, (c_i, l_i)_{i=1}^m\Big)$ be an equilibrium.

Denoting by $\lambda_{i,t}$ and $\mu_{i,t}$ the multipliers associated to the budget constraint of agent i and to the borrowing constraint $l_{i,t} \geq 0$, we obtain the following FOCs for the economy $\mathcal{E}$:

$$\beta_i^t u_i'(c_{i,t}) = \lambda_{i,t} p_t \tag{35}$$

$$\lambda_{i,t} q_t = \lambda_{i,t+1}(q_{t+1} + p_{t+1} F'_{i,t+1}(l_{i,t+1})) + \mu_{i,t+1}, \quad \mu_{i,t+1} l_{i,t+1} = 0. \tag{36}$$

As above, we introduce the dividends of land:

$$\frac{q_t}{p_t} = \gamma_{t+1}\Big(\frac{q_{t+1}}{p_{t+1}} + d_{t+1}\Big)$$

where γ_{t+1} is the discount factor of the economy from date t to date $t+1$:

$$\gamma_{t+1} := \max_{i \in \{1,\dots,m\}} \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}$$

We define the discount factor of the economy from initial date to date t as follows: $Q_0 := 1$ and $Q_t := \prod_{s=1}^{t} \gamma_s$ for any $t \geq 1$. Then, the fundamental value of the land is defined by $FV_0 := \sum_{t=1}^{\infty} Q_t d_t$. We say that land bubbles exist if $q_0/p_0 > FV_0$.

5.1.1 Examples of land bubbles

We now construct equilibria with bubbles.

The economy's fundamentals. Assume that there are two agents A and B with a common utility function $u_A(x) = u_B(x) = ln(x)$ but different non-stationary technologies $F_{A,t}(X) = A_t X, F_{B,t}(X) = B_t X$. For the sake of simplicity, we normalize the supply of land to one: $L = 1$, and we consider alternately null endowments: $e_{A,2t} = e_{B,2t+1} = 0$ for any t.

We need the following conditions to ensure the FOCs and identify the right sequence of discount factors of the economy (γ_t).

$$\beta_A(\frac{\beta_B e_{B,2t}}{1+\beta_B} + A_{2t})(\frac{\beta_A e_{A,2t+1}}{1+\beta_A} + A_{2t+1}) \leq \beta_B \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t+1}}{1+\beta_A} \tag{37}$$

$$\beta_B(\frac{\beta_A e_{A,2t-1}}{1+\beta_A} + B_{2t-1})(\frac{\beta_B e_{B,2t}}{1+\beta} + B_{2t}) \leq \beta_A \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t-1}}{1+\beta_A} \tag{38}$$

$$\beta_A(\frac{\beta_B e_{B,2t}}{1+\beta_B} + A_{2t})(\frac{\beta_A e_{A,2t+1}}{1+\beta_A} + B_{2t+1}) \leq \beta_B \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t+1}}{1+\beta_A} \tag{39}$$

$$\beta_B(\frac{\beta_A e_{A,2t-1}}{1+\beta_A} + B_{2t-1})(\frac{\beta_B e_{B,2t}}{1+\beta} + A_{2t}) \leq \beta_A \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t-1}}{1+\beta_A}. \tag{40}$$

These conditions are not too demanding and are satisfied if, for instance, $\beta_A = \beta_B = \beta$,

$$A_{2t}, B_{2t} < \frac{(1-\beta)e_{B,2t}}{1+\beta} \text{ and } A_{2t+1}, B_{2t+1} < \frac{(1-\beta)e_{A,2t+1}}{1+\beta}$$

for any t.

Equilibrium. Let us construct an equilibrium $\big(p_t, q_t, (c_{i,t}, l_{i,t+1})_{i\in I}\big)_t$ as follows. We normalize the sequence of prices: $p_t = 1$ for any t.

In Appendix 10, we compute the equilibrium allocations.

$$l_{A,2t} = L, l_{A,2t+1} = 0, \quad l_{B,2t} = 0, l_{B,2t+1} = L \tag{41}$$

$$c_{A,2t} = q_{2t}L + A_{2t}L, \quad c_{B,2t} + q_{2t}L = e_{B,2t} \tag{42}$$

$$c_{A,2t+1} + q_{2t+1}L = e_{A,2t+1}, \quad c_{B,2t+1} = q_{2t+1}L + B_{2t+1}L \tag{43}$$

as well the equilibrium prices

$$q_{2t} = \frac{\beta_B}{1+\beta_B}e_{B,2t} \text{ and } q_{2t+1} = \frac{\beta_A}{1+\beta_A}e_{A,2t+1}$$

for any $t \geq 0$.

We find also the land dividends and the discount factors

$$\gamma_{2t} = \frac{\beta_A u'_A(c_{A,2t})}{u'_A(c_{A,2t-1})} \text{ and } \gamma_{2t+1} = \frac{\beta_B u'_B(c_{B,2t+1})}{u'_B(c_{B,2t})} \tag{44}$$

$$d_{2t} = A_{2t} \text{ and } d_{2t+1} = B_{2t+1} \tag{45}$$

for any $t \geq 0$.

According to Proposition 6, land bubbles exist if and only if $\sum_{t=0}^{\infty} \frac{d_t}{q_t} < \infty$, i.e.,

$$\sum_{t=0}^{\infty}\frac{A_{2t}}{e_{B,2t}} + \sum_{t=0}^{\infty}\frac{B_{2t+1}}{e_{A,2t+1}} < \infty$$

Intuition. This condition may be interpreted that land dividends are low with respect to endowments. In other words, the existence of bubbles requires low dividends.

The intuition is straightforward. In the odd periods $(2t+1)$, agent B has no endowments. She wants to smooth consumption over time according to her logarithm utility (which satisfies the Inada conditions), but she cannot transfer her wealth from future to this date.[10]

[10]Because she is prevented from borrowing.

By consequence, she accepts to buy land at a higher price: $q_{2t} \geq e_{B,2t}\beta_B/(1+\beta_B)$, independently on agents' productivity. A lower productivity implies lower dividends and a lower fundamental value of land. As long as dividends tend to zero, the land price remains higher than this fundamental value.

We point our some particular cases of our example.

Example 1 (land bubble with endowment growth)**.** *Consider our example. Assume that* $A_t = B_t = A$ *for any* t*. Then land bubbles exist if and only if*

$$\sum_{t=0}^{\infty} \frac{1}{e_{B,2t}} + \sum_{t=0}^{\infty} \frac{1}{e_{A,2t+1}} < \infty$$

Example 1 illustrates Proposition 8. Thus, under a common stationary production function and $f_i = 0$ for any i, land bubbles may appear if endowments tend to infinity. In this example, we see that land bubbles arise if and only if $\sum_{t=1}^{\infty} 1/q_t < \infty$. By the way, this result also illustrates Corollary 4.

Example 2 (land bubble with collapsing land technologies)**.** *Reconsider our example. If* $e_{A,2t+1} = e_{B,2t} = e > 0$ *for any* t*, then land bubbles emerge if and only if* $\sum_{t=0}^{\infty}(A_{2t}+B_{2t+1}) < \infty$.

This result is also related to Bosi, Le Van and Pham (2015) where they show that physical capital bubbles arise if the sum of capital returns is finite.
Some interesting remarks deserve mention.

1. **Bubble vs *i*-bubble**. Since $\lim_{t\to\infty} \beta_i^t u_i'(c_{i,t})q_t = 0$ for $i = A, B$, there are no i-bubbles. However, land bubbles may occur. In this case, any individual value of land is identical and equal to the equilibrium price but it may exceed the fundamental value of land.

2. ***i*-bubble and borrowing constraints**. In the above example, borrowing constraints of both agents are binding at infinitely many dates while every individual bubbles are ruled out and land bubbles may or may not appear. This shows that the values of (individual) bubbles are not the shadow prices of binding borrowing constraints.

3. **Pure bubble (or fiat money)**. We consider a particular case: If $A_t = B_t = 0$ for every t. In this case, the fundamental value of land is zero and an equilibrium is bubbly if the prices of land are strictly positive in any period ($q_t > 0$ for any t). This bubble is called *pure bubble* by (Tirole, 1985). Our example shows that equilibria with pure bubble may exist in infinite-horizon general equilibrium models.

   In this case, the land in our model can be interpreted as *fiat money* in Bewley (1980), Santos and Woodford (1997), Pascoa, Petrassi and Torres-Martinez (2011) where they provide some examples where the fiat money price is strictly positive. Our contribution concerns the existence of bubble of assets with positive and endogenous dividends.

4. **Land bubbles vs monotonicity of prices**. Corollary 4 points out that, under stationary technologies, the existence of land bubble entails the divergence of land prices to infinity. However, in our example with non-stationary technologies, the land prices are given by

$$q_{2t} = \frac{\beta_B}{1+\beta_B} e_{B,2t} \text{ and } q_{2t+1} = \frac{\beta_A}{1+\beta_A} e_{A,2t+1}$$

and we see that land prices may either increase or decrease or fluctuate over time whenever bubbles exist. Our result generalizes that of Weil (1990) where he gives an example of bubble with decreasing asset prices. His model is a particular case of ours when land gives no longer fruits from some date on: there exists T such that $A_t = B_t = 0$ for any $t \geq T$.

5. **Do the most productive agents produce?** In the above examples, although agents have linear production functions, these functions are different.

   There is a case where the productivity of agent A is higher than that of agent B, i.e., $A_{2t+1} > B_{2t+1}$, but agent A does not produce at date $2t+1$ while agent B produce at this date. For two reasons: (1) agents are prevented from borrowing, (2) agents' endowments change over time. Although A has a higher productivity at date $2t+1$, she has also a higher endowment at this date, but no endowment at date $2t$. So, she may not need to buy land at date $2t$ to produce and transfer wealth from date $2t$ to date $2t+1$. Instead, she sells land at date $2t$ to buy and consume consumption good at date $2t$. Therefore, agent A may not produce at date $2t+1$ even if $A_{2t+1} > B_{2t+1}$.

Remark 14. *Using similar methods, we may construct other examples of bubbles with non-linear production functions, for example $F_{i,t}(x) = A_{i,t} ln(1+x)$ where $A_{i,t} \geq 0$.*

5.1.2 Example of individual land bubbles

The economy's fundamentals. Consider the example in Section 5.1.1. For the sake of simplicity, we assume that $\beta_A = \beta_B =: \beta$.

We add the third agent: agent D. The utility, the rate of time preference, and the technologies of agent D are: $u_D(c) = ln(c), \quad \beta_D = \beta, \quad F_{D,t}(L) = D_t L$.

The endowments $(e_{D,t})_t$ and productivities (D_t) of agents D are defined by

$$\frac{\beta e_{D,t}}{e_{D,t+1}} = \frac{q_t}{q_{t+1} + D_{t+1}} = \gamma_{t+1}$$

where (γ_t) is determined as in Section 5.1.1. We see that such sequences $(e_{D,t})_t$ and (D_t) exist. Indeed, for example, we choose $D_t = d_t$ where d_t is determined as in (45). Then, we choose $(e_{D,t})_t$ such that $\beta e_{D,t} = \gamma_{t+1} e_{D,t+1}$.

Equilibrium: Prices and allocations of agents A and B are as in Section 5.1.1. The allocations of agent D are $c_{D,t} = e_{D,t}$ and $l_{D,t} = 0$ for any t. By using the same argument in Section 5.1.1, it is easy to verify that this system of prices and allocations constitutes an equilibrium.

We observe that agent D does not trade and $\gamma_{D,t} = \beta e_{D,t-1}/e_{D,t} = \gamma_t$ for any t. By consequence, $\lim_{t\to\infty} Q_{D,t} q_t = \lim_{t\to\infty} Q_t q_t > 0$. There is a D - bubble, i.e. the equilibrium price of land is strictly higher than the individual value of land with respect to agent D.

5.2 Land bubbles with short-sales

The economy's fundamentals. Assume that there are two agents A and B with a common utility function $u_A(x) = u_B(x) = ln(x)$ but different non-stationary technologies: $F_{A,t}(X) = A_t X, F_{B,t}(X) = B_t X$ with

$$B_{2t} \geq A_{2t}, \quad A_{2t+1} \geq B_{2t+1}$$

for any t. The supply of land is $L = 1$. Borrowing limits are $f_A = f_B = 1$. For simplicity, we assume that $\beta_A = \beta_B = \beta \in (0, 1)$.

We will find equilibria such that

$$(l_{A,2t}, l_{B,2t}) = (0,1), \quad (l_{A,2t+1}, l_{B,2t+1}) = (1,0) \tag{46}$$

$$a_{A,2t} = \frac{q_{2t} + B_{2t}}{R_{2t}} = -a_{B,2t}, \quad a_{B,2t-1} = \frac{q_{2t-1} + A_{2t-1}}{R_{2t-1}} = -a_{A,2t-1} \quad \forall t \geq 1. \tag{47}$$

It means that at any even (odd) date, agent A (agent B) borrows until her borrowing constraint is binding and buys land.[11] In this case, we have

$$\begin{aligned}
\forall t \geq 0, \quad c_{A,2t} + q_{2t} + a_{A,2t+1} &= e_{A,2t} + R_{2t}a_{A,2t} \\
\forall t \geq 0, \quad c_{A,2t+1} + a_{A,2t+2} &= e_{A,2t+1} \\
c_{B,0} + a_{B,1} &= e_{B,0} + q_0 + B_0 \\
\forall t \geq 1, \quad c_{B,2t} + a_{B,2t+1} &= e_{B,2t} \\
\forall t \geq 0, \quad c_{B,2t+1} + q_{2t+1} + a_{B,2t+2} &= e_{B,2t+1} + R_{2t+1}a_{B,2t+1}
\end{aligned}$$

Since $a_{B,2t+1} > 0$ and $a_{A,2t} > 0$, we have $\mu_{B,2t+1} = \mu_{A,2t} = 0$. Since agent A produces at date $2t+1$ and agent B produces at date $2t$, we have $\eta_{A,2t+1} = \eta_{B,2t} = 0$.

The consumption good is taken as numéraie $p_t = 1$. We have to find land prices and interest rates satisfying first order and transversality conditions.

Transversality conditions (16) write $\lim_{t\to\infty} \beta^t u_i'(c_{i,t})(q_t l_{i,t+1} + a_{i,t+1}) = 0$ for any t, which is equivalent to

$$\lim_{t\to\infty} \frac{\beta^{2t}(q_{2t} - \frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}})}{e_{A,2t} + (q_{2t} + B_{2t}) - q_{2t} + \frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}}} = 0, \quad \lim_{t\to\infty} \frac{\beta^{2t+1}\frac{q_{2t+2}+B_{2t+2}}{R_{2t+2}}}{e_{A,2t+1} - \frac{q_{2t+2}+B_{2t+2}}{R_{2t+2}}} = 0$$

$$\lim_{t\to\infty} \frac{\beta^{2t}\frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}}}{e_{B,2t} - \frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}}} = 0, \quad \lim_{t\to\infty} \frac{\beta^{2t+1}(q_{2t+1} - \frac{q_{2t+2}+B_{2t+2}}{R_{2t+2}})}{e_{B,2t+1} + (q_{2t+1} + A_{2t+1}) - q_{2t+1} + \frac{q_{2t+2}+B_{2t+2}}{R_{2t+2}}} = 0.$$

FOCs can be rewritten as

$$1 = \beta \frac{e_{B,0} + q_0 + B_0 - \frac{q_1+A_1}{R_1}}{e_{B,1} + (q_1 + A_1) - q_1 + \frac{q_2+B_2}{R_2}} R_1 > \beta \frac{e_{A,0} - q_0 + \frac{q_1+A_1}{R_1}}{e_{A,1} - \frac{q_2+B_2}{R_2}} R_1 \tag{48}$$

$$\begin{aligned}
1 &= \beta \frac{e_{B,2t} - \frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}}}{e_{B,2t+1} + (q_{2t+1} + A_{2t+1}) - q_{2t+1} + \frac{q_{2t+2}+B_{2t+2}}{R_{2t+2}}} R_{2t+1} \\
&> \beta \frac{e_{A,2t} + (q_{2t} + B_{2t}) - q_{2t} + \frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}}}{e_{A,2t+1} - \frac{q_{2t+2}+B_{2t+2}}{R_{2t+2}}} R_{2t+1}
\end{aligned} \tag{49}$$

$$\begin{aligned}
1 &= \beta \frac{e_{A,2t-1} - \frac{q_{2t}+B_{2t}}{R_{2t}}}{e_{A,2t} + (q_{2t} + B_{2t}) - q_{2t} + \frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}}} R_{2t} \\
&> \beta \frac{e_{B,2t-1} + (q_{2t-1} + A_{2t-1}) - q_{2t-1} + \frac{q_{2t}+B_{2t}}{R_{2t}}}{e_{B,2t} - \frac{q_{2t+1}+A_{2t+1}}{R_{2t+1}}} R_{2t}.
\end{aligned} \tag{50}$$

Condition (48) means that $1 = \gamma_{B,1}R_1 > \gamma_{A,1}R_1$. Condition (49) means that $1 = \gamma_{B,2t+1}R_{2t+1} > \gamma_{A,2t+1}R_{2t+1}$ while condition (50) means that $1 = \gamma_{A,2t}R_{2t} > \gamma_{B,2t}R_{2t}$. This implies that $\gamma_{2t} = \gamma_{A,2t}$ and $\gamma_{2t+1} = \gamma_{B,2t+1}$.

Since $f_A = f_B = 1$, according to Lemma 4, dividends are determined by

$$d_{2t} = B_{2t} \text{ and } d_{2t+1} = A_{2t+1}. \tag{51}$$

[11] We need this because Corollary 6 indicates that bubbles only exist if borrowing constraints of agents are frequently binding.

Remark 15 (endogenous dividends). *Comparing our example in this section and that in Section 5.1.1, the technologies of two agents A and B do not change but land dividends change (see (45) and (51)). This difference is from the fact that land dividends are endogenous defined.*

Applying Proposition 2 allows us obtain the following result.

Example 3 (land bubbles with short-sales). *Let endowments such that*[12]

$$
\begin{aligned}
e_{B,2t-1} = e_{A,2t} &= 0 \quad \forall t \geq 1 \\
\frac{e_{B,2t}e_{A,2t+1}}{(1+\beta)^2} &> \Big(\frac{\beta}{1+\beta}e_{B,2t} + B_{2t}\Big)\Big(\frac{\beta}{1+\beta}e_{A,2t+1} + A_{2t+1}\Big) \quad \forall t \geq 0 \\
\frac{e_{B,2t}e_{A,2t-1}}{(1+\beta)^2} &> \Big(\frac{\beta}{1+\beta}e_{B,2t} + B_{2t}\Big)\Big(\frac{\beta}{1+\beta}e_{A,2t-1} + A_{2t-1}\Big) \quad \forall t \geq 1
\end{aligned}
$$

Equilibrium prices are determined as follows, for any $t \geq 1$,

$$
\begin{aligned}
p_t &= 1 \quad \forall t \\
q_0 &= \beta(e_{B,0} + B_0), \quad q_{2t} = \frac{\beta}{1+\beta}e_{B,2t}, \quad q_{2t-1} = \frac{\beta}{1+\beta}e_{A,2t-1}, \\
R_{2t} &= \frac{q_{2t} + B_{2t}}{q_{2t-1}}, \quad R_{2t-1} = \frac{q_{2t-1} + A_{2t-1}}{q_{2t-2}}.
\end{aligned}
$$

It is easy to check that under these specifications, the above first order and transversality conditions are satisfied. According to Proposition 6, land bubbles exist if and only if

$$
\sum_{t=1}^{\infty} \frac{B_{2t}}{e_{B,2t}} + \sum_{t=0}^{\infty} \frac{A_{2t+1}}{e_{A,2t+1}} < \infty. \tag{52}
$$

As in economies without-short sales, bubbles may occur if endowments growth without bound and/or TFP tends to zero.

The intuition *of our example. Look at the economy at date $2t$. Agent B knows that she will not have endowment at date $2t+1$: $e_{B,2t+1} = 0$, and hence she wants to transfer her wealth from date $2t$ to date $2t+1$ (she saves at date $2t$). Therefore, she may accept to buy land with a high price or buy financial asset with low interest rates. The same argument applies for the agent A at date $2t+1$. Therefore, the price of land may be higher than its fundamental value or equivalently the bubble component $\lim\limits_{t\to\infty} \frac{q_t}{R_1R_2\ldots R_t}$ may be strictly positive.*

Some observations should be mentioned.

1. Look at the economy at date $2t$. Agent B anticipates that the real return of the financial asset at date $2t+1$ is $R_{2t+1} = \frac{q_{2t+1}+A_{2t+1}}{q_{2t}}$ is strictly higher than the real return $\frac{q_{2t+1}+B_{2t+1}}{q_{2t}}$ of agent B if she produces. So, agent B does not buy land at this date, she buys the financial asset and sells all her land. By contrast, at equilibrium agent A borrows and produces at this date.[13]

2. **With vs without short-sales**. In Examples without short-sale in Section 5.1.1, agents transfer there wealth from one date to the next date by the unique way: buying land. However, in Example 3, they do so by investing in the financial market or buying land. Thank to the financial market, land is used by the most productive agent in Example 3. This is not true when agents are prevented from borrowing as showed in Section 5.1.1.

[12] These conditions are similar to (37, 38, 39, 40) in our example of bubble in models without short-sales.

[13] In fact, for agent A the financial return equals the land return if she produces. However, only the solution $l_{A,2t+1} = L$ and $a_{A,2t+1} < 0$ clears both land and financial markets.

6 Conclusion

We have considered dynamic general equilibrium models with heterogeneous agents and financial markets and found that, when the financial market is "good enough" (borrowing limits are equal to one), agents with high productivity produce and borrow until their borrowing constraints become binding, while agents with low productivity lend and do not produce. Differently from standard capital accumulation models à la Ramsey, the most patient may not hold the entire stock of land in the long run.

Our paper has provided an approach to the evaluation of land by introducing *endogenous land dividends*, and studied the occurrence and the general equilibrium impact of land bubbles. Under standard assumptions, land bubbles are ruled out if borrowing limits equal one and the production functions are stationary. When financial frictions take place and agents' wealth dynamics are heterogeneous, land bubbles may arise.

After having introduced the concept of i-bubble based on interest rates expected by agent i, we have proved that if the land value expected by each individual equals the fundamental value of land, then there is neither bubble nor i-bubble. A number of examples of (individual) bubbles are provided in economies with and without short-sales.

Our analysis suggests that the existence of asset bubbles depends not only on the asset structure but also on the discount factors we choose to evaluate the asset fundamental value. Our approach can be used to evaluate other kinds of asset or input such as house or physical capital.

Nevertheless, our economy remains deterministic: it would be interesting to extend our analysis for economies with uncertainty (Santos and Woodford, 1997; Pascoa, Petrassi and Torres-Martinez, 2011).

7 Appendix: Proofs for Section 2.2

Proof of part 1 of Proposition 2. The first order conditions are standard.

Let us prove the transversality condition. Denote $x_i := (l_i, a_i) = (l_{i,t}, a_{i,t})_t$. We say that x_i is feasible if, for every t, we have $l_{i,t} \geq 0$ and

$$R_t a_{i,t} \geq -f_i(q_t l_{i,t} + p_t F_i(l_{i,t}))$$
$$q_t l_{i,t+1} + p_t a_{i,t+1} \leq p_t e_{i,t} + q_t l_{i,t} + p_t F_i(l_{i,t}) + R_t a_{i,t}.$$

We claim that: if x_i is feasible, then $(x_{i,0}, \ldots, x_{i,t}, \lambda x_{i,t+1}, \lambda x_{i,t+2}, \ldots)$ is also feasible for each $t \geq 1$ and $\lambda \in [0, 1]$.

We have to prove that:

$$q_t \lambda l_{i,t+1} + p_t \lambda a_{i,t+1} \leq p_t e_{i,t} + q_t l_{i,t} + p_t F_i(l_{i,t}) + R_t a_{i,t} \tag{53}$$

and

$$\lambda q_s l_{i,s+1} + \lambda p_s a_{i,s+1} \leq p_s e_{i,s} + \lambda q_s l_{i,s} + p_s F_i(\lambda l_{i,s}) + \lambda R_s a_{i,t} \tag{54}$$
$$\lambda R_s a_{i,s+1} + f_i(\lambda q_{s+1} l_{i,s+1} + p_{s+1} F_i(\lambda l_{i,s+1})) \geq 0 \tag{55}$$

for each $s \geq t$.

(54) and (55) are proved by using the fact that $F_i(\lambda x) \geq \lambda F_i(x)$ for every $\lambda \in [0, 1]$.

(53) is satisfied if $q_t l_{i,t+1} + p_t a_{i,t+1} < 0$. If $q_t l_{i,t+1} + p_t a_{i,t+1} \geq 0$, we have

$$q_t \lambda l_{i,t+1} + p_t \lambda a_{i,t+1} \leq q_t l_{i,t+1} + p_t a_{i,t+1} \leq p_t e_{i,t} + q_t l_{i,t} + p_t F_i(l_{i,t}) + R_t a_{i,t}.$$

By using the same argument in Theorem 2.1 in Kamihigashi (2002),[14] we obtain that $\limsup_{t\to\infty} \lambda_{i,t}(q_t l_{i,t+1} + p_t a_{i,t+1}) \leq 0$.

According to FOCs, we now have

$$\lambda_{i,t}(p_t c_{i,t} + q_t l_{i,t+1} + p_t a_{i,t+1}) = \lambda_{i,t}(p_t e_{i,t} + q_t l_{i,t} + p_t F_i(l_{i,t}) + R_t a_{i,t}) \tag{56}$$
$$\lambda_{i,t} p_t a_{i,t+1} = (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} a_{i,t+1} \tag{57}$$
$$\lambda_{i,t} q_t l_{i,t+1} = (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})) l_{i,t+1} \tag{58}$$
$$\mu_{i,t+1}\Big(R_{t+1} a_{i,t+1} + f_i(q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1}))\Big) = 0. \tag{59}$$

(57) and (58) imply that

$$\begin{aligned}
&\lambda_{i,t}(q_t l_{i,t+1} + p_t a_{i,t+1}) \\
&= (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})) l_{i,t+1} + (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} a_{i,t+1} \\
&= \lambda_{i,t+1} R_{t+1} a_{i,t+1} + \lambda_{i,t+1}\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1} \\
&+ \mu_{i,t+1}\Big(R_{t+1} a_{i,t+1} + f_i\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1}\Big)
\end{aligned}$$

Therefore, by combining this with (59), we get that

$$\begin{aligned}
&\lambda_{i,t}(q_t l_{i,t+1} + p_t a_{i,t+1}) - \lambda_{i,t+1} R_{t+1} a_{i,t+1} - \lambda_{i,t+1}\big(q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1})\big) \\
&= \lambda_{i,t+1}\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1} + \mu_{i,t+1}\Big(R_{t+1} a_{i,t+1} + f_i\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1}\Big) \\
&\quad - \lambda_{i,t+1}\big(q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1})\big) &(60)\\
&= -\lambda_{i,t+1} p_{t+1}(F_i(l_{i,t+1}) - l_{i,t+1} F_i'(l_{i,t+1})) + \mu_{i,t+1}\Big(R_{t+1} a_{i,t+1} + f_i\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1}\Big) \\
&- \mu_{i,t+1}\Big(R_{t+1} a_{i,t+1} + f_i\big(q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1})\big)\Big) &(61)\\
&= -\lambda_{i,t+1} p_{t+1}(F_i(l_{i,t+1}) - l_{i,t+1} F_i'(l_{i,t+1})) - f_i \mu_{i,t+1} p_{t+1}(F_i(l_{i,t+1}) - l_{i,t+1} F_i'(l_{i,t+1})) &(62)
\end{aligned}$$

By summing (56) from $t = 0$ to T, and then using (60), we obtain that

$$\begin{aligned}
\sum_{t=0}^{T} \lambda_{i,t} p_t c_{i,t} + \lambda_{i,T}(q_T l_{i,T+1} + p_T a_{i,T+1}) = &\sum_{t=0}^{T} \lambda_{i,t} p_t e_{i,t} + \lambda_{i,0}(q_0 l_{i,0} + p_t F_i(l_{i,0}) + R_0 a_{i,0}) \\
&+ \sum_{t=1}^{T} p_t(\lambda_{i,t} + f_i \mu_{i,t})(F_i(l_{i,t}) - l_{i,t} F_i'(l_{i,t})). \qquad (63)
\end{aligned}$$

Under Assumption (5), the utility of agent i is finite, thus we have

$$\sum_{t=0}^{\infty} \lambda_{i,t} p_t c_{i,t} = \sum_{t=0}^{\infty} \beta_i^t u_i'(c_{i,t}) c_{i,t} \leq \sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) < \infty. \tag{64}$$

Combining this with $\limsup_{t\to\infty} \lambda_{i,t}(q_t l_{i,t+1} + p_t a_{i,t+1}) \leq 0$ and (63), we obtain that there exists the following sum

$$\sum_{t=0}^{\infty} \lambda_{i,t} p_t e_{i,t} + \sum_{t=1}^{\infty} p_t(\lambda_{i,t} + f_i \mu_{i,t})(F_i(l_{i,t}) - l_{i,t} F_i'(l_{i,t})) < \infty.$$

[14]Kamihigashi (2002) only considers positive allocations while $a_{i,t}$ may be negative in our model.

We now use (63) to get that $\lim_{t\to\infty} \lambda_{i,t}(q_t l_{i,t+1} + p_t a_{i,t+1})$ exists and it is non positive.

We again use (56) and note that $q_t l_{i,t} + p_t F_i(l_{i,t}) + R_t a_{i,t} \geq 0$ (because of borrowing constraint) to obtain that $\liminf_{t\to\infty} \lambda_{i,t}(p_t c_{i,t} + q_t l_{i,t+1} + p_t a_{i,t+1}) \geq 0$.

(64) implies that $\lim_{t\to\infty} \lambda_{i,t} p_t c_{i,t} = 0$. As a result, we get $\liminf_{t\to\infty} \lambda_{i,t}(q_t l_{i,t+1} + p_t a_{i,t+1}) \geq 0$. Therefore, we have $\lim_{t\to\infty} \lambda_{i,t}(q_t l_{i,t+1} + p_t a_{i,t+1}) = 0$ and then

$$\infty > \sum_{t=0}^{\infty} \lambda_{i,t} p_t c_{i,t} = \sum_{t=0}^{\infty} \lambda_{i,t} p_t e_{i,t} + \sum_{t=1}^{\infty} p_t(\lambda_{i,t} + f_i \mu_{i,t})(F_i(l_{i,t}) - l_{i,t} F_i'(l_{i,t})) \\ + \lambda_{i,0}(q_0 l_{i,0} + p_t F_i(l_{i,0}) + R_0 a_{i,0}). \tag{65}$$

□

Proof of part 2 of Proposition 2. Without loss of generality, we can assume $p_t = 1$ for any t.

Let $(c_i', a_i', l_i') \geq 0$ be a plan satisfying all budget and borrowing constraints and $l_{i,0}' - l_{i,0} = 0 = a_{i,0}' = a_{i,0}$. We have

$$\sum_{t=0}^{T} \beta_i^t (u_i(c_{i,t}) - u_i(c_{i,t}')) \geq \sum_{t=0}^{T} \beta_i^t u_i'(c_{i,t})(c_{i,t} - c_{i,t}') = \sum_{t=0}^{T} \lambda_{i,t} p_t (c_{i,t} - c_{i,t}')$$

According to FOCs, we now have

$$\lambda_{i,t}(p_t c_{i,t}' + q_t l_{i,t+1}' + p_t a_{i,t+1}') = \lambda_{i,t}(p_t e_{i,t} + q_t l_{i,t}' + p_t F_i(l_{i,t}') + R_t a_{i,t}') \tag{66}$$
$$\lambda_{i,t} p_t a_{i,t+1}' = (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} a_{i,t+1}' \tag{67}$$
$$\lambda_{i,t} q_t l_{i,t+1}' = (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})) l_{i,t+1}' + \eta_{i,t+1} l_{i,t+1}' \tag{68}$$
$$\mu_{i,t+1}\Big(R_{t+1} a_{i,t+1} + f_i(q_{t+1} l_{i,t+1} + p_{t+1} F_i(l_{i,t+1}))\Big) = 0. \tag{69}$$

(67) and (68) imply that

$$\lambda_{i,t}(q_t l_{i,t+1}' + p_t a_{i,t+1}') \tag{70}$$
$$= (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})) l_{i,t+1}' + \eta_{i,t+1} l_{i,t+1}' + (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} a_{i,t+1}'$$
$$= \lambda_{i,t+1} R_{t+1} a_{i,t+1}' + \lambda_{i,t+1}\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1}' + \eta_{i,t+1} l_{i,t+1}' \\ + \mu_{i,t+1}\Big(R_{t+1} a_{i,t+1}' + f_i\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1}'\Big). \tag{71}$$

Therefore, by combining this with (69), we get that

$$\lambda_{i,t}(q_t l_{i,t+1}' + p_t a_{i,t+1}') - \lambda_{i,t+1} R_{t+1} a_{i,t+1}' - \lambda_{i,t+1}\big(q_{t+1} l_{i,t+1}' + p_{t+1} F_i(l_{i,t+1}')\big) \\ = \lambda_{i,t+1}\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1}' + \eta_{i,t+1} l_{i,t+1}' + \mu_{i,t+1}\Big(R_{t+1} a_{i,t+1}' + f_i\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1}'\Big) \\ - \lambda_{i,t+1}\big(q_{t+1} l_{i,t+1}' + p_{t+1} F_i(l_{i,t+1}')\big) \\ = -\lambda_{i,t+1} p_{t+1}(F_i(l_{i,t+1}') - l_{i,t+1}' F_i'(l_{i,t+1})) + \eta_{i,t+1} l_{i,t+1}' \\ + \mu_{i,t+1}\Big(R_{t+1} a_{i,t+1}' + f_i\big(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})\big) l_{i,t+1}'\Big). \tag{72}$$

According to budget constraints, we have

$$\lambda_{i,t+1} p_{t+1}(e_{i,t+1} - c_{i,t+1}') + \lambda_{i,t}(q_t l_{i,t+1}' + p_t a_{i,t+1}') - \lambda_{i,t+1}(q_{t+1} l_{i,t+2}' + p_{t+1} a_{i,t+2}') \\ = \lambda_{i,t}(q_t l_{i,t+1}' + p_t a_{i,t+1}') - \lambda_{i,t+1} R_{t+1} a_{i,t+1}' - \lambda_{i,t+1}\big(q_{t+1} l_{i,t+1}' + p_{t+1} F_i(l_{i,t+1}')\big) \tag{73}$$
$$\lambda_{i,0} p_0(e_{i,0} - c_{i,0}') - \lambda_{i,0}(q_0 l_{i,1}' + p_0 a_{i,1}') + \lambda_{i,0}\big(q_0 l_{i,0}' + p_0 F_i(l_{i,0}')\big) = 0. \tag{74}$$

By summing these constraints and using (72), we obtain that

$$\Big[\sum_{t=0}^{T}\lambda_{i,t}p_t(e_{i,t}-c'_{i,t})\Big]-\lambda_{i,T}(q_T l'_{i,T+1}+p_T a'_{i,T+1})+\lambda_{i,0}\big(q_0 l'_{i,0}+p_0F_i(l'_{i,0})\big)$$
$$=\sum_{t=1}^{T}\Big[-\lambda_{i,t}p_{i,t}\big(F_i(l'_{i,t})-l'_{i,t}F'_i(l_{i,t})\big)+\eta_{i,t}l'_{i,t}+\mu_{i,t}\big(R_t a'_{i,t}+f_i\big(q_t+p_tF'_i(l_{i,t})\big)l'_{i,t}\Big]$$

Since this is satisfied for any feasible allocation (c'_i, a'_i, l'_i), this also holds for the allocation (c_i, a_i, l_i). Consequently, we get

$$\Big[\sum_{t=0}^{T}\lambda_{i,t}p_t(c_{i,t}-c'_{i,t})\Big]$$
$$=\lambda_{i,T}(q_T l'_{i,T+1}+p_T a'_{i,T+1})-\lambda_{i,T}(q_T l_{i,T+1}+p_T a_{i,T+1})$$
$$+\sum_{t=1}^{T}\Big[-\lambda_{i,t}p_{i,t}\big(F_i(l'_{i,t})-l'_{i,t}F'_i(l_{i,t})\big)+\eta_{i,t}l'_{i,t}+\mu_{i,t}\big(R_t a'_{i,t}+f_i\big(q_t+p_tF'_i(l_{i,t})\big)l'_{i,t}\Big]$$
$$-\sum_{t=1}^{T}\Big[-\lambda_{i,t}p_{i,t}\big(F_i(l_{i,t})-l_{i,t}F'_i(l_{i,t})\big)+\eta_{i,t}l_{i,t}+\mu_{i,t}\big(R_t a_{i,t}+f_i\big(q_t+p_tF'_i(l_{i,t})\big)l_{i,t}\Big]$$
$$\geq -\lambda_{i,T}(q_T l_{i,T+1}+p_T a_{i,T+1})+\sum_{t=1}^{T}\lambda_{i,t}p_{i,t}\Big[F_i(l_{i,t})-F_i(l'_{i,t})-(l_{i,t}-l'_{i,t})F'(l_{i,t})\Big]$$
$$-\sum_{t=1}^{T}\mu_{i,t}\big(R_t a_{i,t}+f_i\big(q_t+p_tF'_i(l_{i,t})\big)l_{i,t}$$

Since F_i is concave, it is easy to see that

$$F_i(l_{i,t})-F_i(l'_{i,t})\geq (l_{i,t}-l'_{i,t})F'(l_{i,t})$$
$$\mu_{i,t}\big(R_t a_{i,t}+f_i\big(q_t+p_tF'_i(l_{i,t})\big)l_{i,t}\leq \mu_{i,t}\big(R_t a_{i,t}+f_i\big(q_t l_{i,t}+p_tF_i(l_{i,t})\big)=0.$$

Thus, we obtain

$$\sum_{t=0}^{T}\beta_i^t(u_i(c_{i,t})-u_i(c'_{i,t}))\geq\sum_{t=0}^{T}\lambda_{i,t}p_t(c_{i,t}-c'_{i,t})\geq -\lambda_{i,T}(q_t l_{i,T+1}+p_T a_{i,T+1})$$

By combining this with (16) and the fact that $\sum_{t=0}^{\infty}\beta_i^t u_i(c_{i,t})<\infty$, we conclude the optimality of (c_i, a_i, l_i). □

Proof of Corollary 1. Assume that there exists $\lim_{t\to\infty}\Big(Q_t a_{i,t+1}+f_iQ_{t+1}\big[\frac{q_{t+1}}{p_{t+1}}l_{i,t+1}+F_i(l_{i,t+1})\big]\Big)>0$. Hence, there exists a date $T\geq 1$ such that borrowing constraint (4) is not binding for every $t\geq T$. Therefore, $\lambda_{i,t}p_t=\lambda_{i,t+1}R_{t+1}$ for every $t\geq T$. By consequence, there exists a constant $C_i\in(0,\infty)$ such that $Q_t=C_i\lambda_{i,t}p_t$ for every $t\geq T$. According to transversality condition (16), we get $\lim_{t\to\infty}Q_t\big(a_{i,t+1}+\frac{q_t}{p_t}l_{i,t+1}\big)=0$.

By combining (17) and the fact that $Q_t=C_i\lambda_{i,t}p_t$ for every $t\geq T$, we obtain $\lim_{t\to\infty}Q_tc_{i,t}=\lim_{t\to\infty}Q_te_{i,t}=0$. Therefore, by using budget constraints, we get

$$\lim_{t\to\infty}Q_t\big(\frac{R_t}{p_t}a_{i,t}+\frac{q_t}{p_t}l_{i,t}+F_i(l_{i,t})\big)=0.$$

Since $f_i \in [0,1]$ and $Q_t \frac{R_t}{p_t} = Q_{t-1}$, we obtain the statement (b).

(19) is proved by using the same argument. $\square$

8 Appendix: Proofs for Section 3

Proof of Lemma 3. According to (13), we obtain $\dfrac{q_t}{p_t} \geq \gamma_{t+1}\big(\dfrac{q_{t+1}}{p_{t+1}} + \underline{d}_{t+1}\big)$.

We prove the second inequality. We see that there exists an agent, say i, such that $l_{i,t+1} > 0$. Thus, $\eta_{i,t+1} = 0$. Therefore, we have

$$\begin{aligned}\lambda_{i,t} q_t &= (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + p_{t+1} F_i'(l_{i,t+1}))\\ &\leq (\lambda_{i,t+1} + \mu_{i,t+1})(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})) \leq \frac{\lambda_{i,t} p_t}{R_{t+1}}(q_{t+1} + p_{t+1}\bar{d}_{t+1}). \quad (75)\end{aligned}$$

By combining with (18), we get the second inequality in (20).

We now prove (21). According to FOCs, we get

$$\begin{aligned}q_t &= \frac{\lambda_{i,t+1} + f_i \mu_{i,t+1}}{\lambda_{i,t+1} + \mu_{i,t+1}} \frac{\lambda_{i,t+1} + \mu_{i,t+1}}{\lambda_{i,t}}(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})) + \frac{\eta_{i,t+1}}{\lambda_{i,t}} \quad (76)\\ &= \frac{\lambda_{i,t+1} + f_i \mu_{i,t+1}}{\lambda_{i,t+1} + \mu_{i,t+1}} \frac{p_t}{R_{t+1}}(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})) + \frac{\eta_{i,t+1}}{\lambda_{i,t}} \quad (77)\\ &\geq f_i \frac{p_t}{R_{t+1}}(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})). \quad (78)\end{aligned}$$

Therefore, we obtain (21). $\square$

Proof of Lemma 4. According to (20), we obtain $d_{t+1} \leq \bar{d}_{t+1}$.

Since $f_i = 1$ for any i or (4) is not binding for any i, we always have $\mu_{i,t+1} = f_i \mu_{i,t+1}$ for every i. So, we get

$$\frac{q_t}{p_t} = \gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\big) + \frac{\eta_{i,t+1}}{\lambda_{i,t} p_t} \geq \gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\big)$$

for any i. Therefore $d_{t+1} \geq \bar{d}_{t+1}$. As a result, we have $d_{t+1} = \bar{d}_{t+1}$. $\square$

Proof of Proposition 3. Since $l_{i,t+1} > 0$ at equilibrium, and then $\eta_{i,t+1} = 0$. By consequence, we obtain, for every i, t,

$$\begin{aligned}\lambda_{i,t} p_t &= (\lambda_{i,t+1} + \mu_{i,t+1}) R_{t+1} \quad (79)\\ \lambda_{i,t} q_t &= (\lambda_{i,t+1} + f_i \mu_{i,t+1})(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})). \quad (80)\end{aligned}$$

We see that, for every i, t,

$$\begin{aligned}q_t = \frac{\lambda_{i,t+1} + f_i \mu_{i,t+1}}{\lambda_{i,t}}(q_{t+1} + p_{t+1} F_i'(l_{i,t+1})) &\leq \frac{\lambda_{i,t+1} + \mu_{i,t+1}}{\lambda_{i,t}}(q_{t+1} + p_{t+1} F_i'(l_{i,t+1}))\\ &= \gamma_{t+1}(q_{t+1} + p_{t+1} F_i'(l_{i,t+1}))\end{aligned}$$

Therefore, we obtain that $\dfrac{q_t}{p_t} \leq \gamma_{t+1}\big(\dfrac{q_{t+1}}{p_{t+1}} + \underline{d}_{t+1})\big)$. By combining with (20), we have

$$\frac{q_t}{p_t} = \gamma_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + \underline{d}_{t+1}\big).$$

As a result, we get that $d_{t+1} = \underline{d}_{t+1}$. $\square$

Proof of Proposition 4. According to FOCs, we obtain

$$1 = \frac{\lambda_{i,t+1} + f_i\mu_{i,t+1}}{\lambda_{i,t+1} + \mu_{i,t+1}} \frac{q_{t+1} + p_{t+1}F_i'(l_{i,t+1})}{q_{t+1} + p_{t+1}d_{t+1}} + \frac{\eta_{i,t+1}}{\lambda_{i,t}q_t} \tag{81}$$

If $l_{i,t+1} > 0$, then $\eta_{i,t+1} = 0$. By combining (81) with $f_i \leq 1$, we get $d_{t+1} \leq F_i'(l_{i,t+1})$.

We now assume that $d_{t+1} < F_i'(l_{i,t+1})$. If (4) is not binding, we have $\mu_{i,t+1} = 0$ which implies that $d_{t+1} \geq F_i'(l_{i,t+1})$, contradiction. □

Proof of Proposition 5. Since $f_j = 1$, (13) implies that

$$\frac{q_t}{q_{t+1} + p_{t+1}F_j'(l_{j,t+1})} \geq \frac{\lambda_{j,t+1} + \mu_{j,t+1}}{\lambda_{j,t}} = \frac{p_t}{R_{t+1}}.$$

Assume that $l_{i,t+1} > 0$, we have $\eta_{i,t+1} = 0$ which implies that

$$\frac{q_t}{q_{t+1} + p_{t+1}F_i'(l_{i,t+1})} = \frac{\lambda_{i,t+1} + f_i\mu_{i,t+1}}{\lambda_{i,t}} \leq \frac{p_t}{R_{t+1}} \leq \frac{q_t}{q_{t+1} + p_{t+1}F_j'(l_{j,t+1})}.$$

Therefore, $F_j'(L) \leq F_j'(l_{j,t+1}) \leq F_i'(l_{i,t+1}) < F_i'(0)$, contradiction. □

Proof of Lemma 5. Let $\big((c_i, l_i, a_i), (c_j, l_j, a_j), p, q, R\big)$ be a steady state equilibrium. According to Proposition 2, we rewrite the system (11, 12, 13, 14, 15)

$$\begin{aligned}
&\beta_i^t u_i'(c_{i,t}) = \lambda_{i,t}p_t\\
&1 = \frac{R_{t+1}}{p_{t+1}}\Big(\frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} + \frac{\mu_{i,t+1}p_{t+1}}{\lambda_{i,t}p_t}\Big)\\
&\frac{q_t}{p_t} = \Big(\frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} + f_i\frac{\mu_{i,t+1}p_{t+1}}{\lambda_{i,t}p_t}\Big)\Big(\frac{q_{t+1}}{p_{t+1}} + F_i'(l_{i,t+1})\Big) + \frac{\eta_{i,t+1}}{\lambda_{i,t}p_t}\\
&\eta_{i,t+1}l_{i,t+1} = 0\\
&\mu_{i,t+1}\Big(R_{t+1}a_{i,t+1} + f_i\big[q_{t+1}l_{i,t+1} + p_{t+1}F_i(l_{i,t+1})\big]\Big) = 0
\end{aligned}$$

Let us denote $x_{i,t} := \dfrac{\mu_{i,t+1}p_{t+1}}{\lambda_{i,t}p_t}$, $\sigma_{i,t} := \dfrac{\eta_{i,t+1}}{\lambda_{i,t}p_t}$.

At steady state, we have

$$\begin{aligned}
1 &= \frac{R}{p}(\beta_i + x_i)\\
\frac{q}{p} &= (\beta_i + f_ix_i)\Big(\frac{q}{p} + F_i'(l_i)\Big) + \sigma_i.
\end{aligned}$$

Since $\beta_i < \beta_j$, we have $x_i > x_j$, which implies that $x_i > 0$. Therefore, we obtain

$$Ra_i + f_i\big[ql_i + pF_i(l_i)\big] = 0.$$

Hence, $a_i < 0$ and then $a_j > 0$ which implies that $x_j = 0$. The impatient agent borrows from the patient agent.

We consider the case where $F_i(l_i) = A_il_i^\alpha, F_j(l_i) = A_jl_j^\alpha$. Then $F_h'(l_h) = \alpha A_hl_h^{\alpha-1}$ for each $h = i, j$. In this case, we have $l_i, l_j > 0$, hence $\sigma_i = \sigma_j = 0$.

We see that $a_i < 0$, which implies that $a_j > 0$. Hence, $x_j = 0$. The asset price is $\dfrac{R}{p} = \dfrac{1}{\beta_j} = 1 + r$, where r is the real interest rate. We have $\dfrac{q}{p}\Big(\frac{1}{\beta_j} - 1\Big) = F_i'(l_i) = \alpha A_j l_j^{\alpha-1}$, therefore

$$l_j = \Big(\frac{\alpha A_j}{\frac{1}{\beta_j} - 1}\frac{p}{q}\Big)^{\frac{1}{1-\alpha}}.$$

Since $\beta_i + x_i = \beta_j + x_j$, we get $x_i = \beta_j - \beta_i$. By consequence, we can compute

$$l_i = \Big(\frac{\alpha A_i}{\frac{1}{\beta_i + f_i(\beta_j - \beta_i)} - 1}\frac{p}{q}\Big)^{\frac{1}{1-\alpha}}.$$

Using $l_i + l_j = L$, we can compute the price of land

$$\Big(\frac{q}{p}\Big)^{\frac{1}{1-\alpha}} L = \Big(\frac{\alpha A_i}{\frac{1}{\beta_i + f_i(\beta_j - \beta_i)} - 1}\Big)^{\frac{1}{1-\alpha}} + \Big(\frac{\alpha A_j}{\frac{1}{\beta_j} - 1}\Big)^{\frac{1}{1-\alpha}}.$$

□

9 Appendix: Proofs for Section 4

Proof of Proposition 6. According to (28), it is easy to see that (i) is equivalent to (ii). We now prove that (ii) and (iii) are equivalent.

According to (27), we get that

$$\frac{q_0}{p_0} = Q_T \frac{q_T}{p_T} \prod_{t=1}^{T} (1 + \frac{p_t d_t}{q_t}).$$

Since $\frac{q_0}{p_0} > 0$, we see that $\lim_{t\to+\infty} Q_t \frac{q_t}{p_t} > 0$ if and only if $\lim_{t\to\infty} \prod_{t=1}^{T}(1 + \frac{p_t d_t}{q_t}) < \infty$. It is easy to prove that this condition is equivalent to $\sum_{t=1}^{\infty} \dfrac{p_t d_t}{q_t} < +\infty$. □

Proof of Proposition 7. Assume that $Q_t/Q_{i,t}$ is uniformly bounded from above. According to Proposition 2, we have $\lim_{t\to\infty} Q_{i,t}(\frac{q_t}{p_t} l_{i,t+1} + a_{i,t+1}) = 0$, therefore

$$\lim_{t\to\infty} Q_t(\frac{q_t}{p_t} l_{i,t+1} + a_{i,t+1}) = \lim_{t\to\infty} \Big(\frac{Q_t}{Q_{i,t}}\Big)\Big(Q_{i,t}(\frac{q_t}{p_t} l_{i,t+1} + a_{i,t+1})\Big) = 0$$

for any i. Note that $\sum_i l_{i,t+1} = L > 0$ and $\sum_i a_{i,t+1} = 0$ for any t, we obtain that $\lim_{t\to\infty} Q_t \frac{q_t}{p_t} = 0$ □

Proof of Corollary 5. Since $\mu_{i,t+1} = 0$ for every $t \geq T$, we have $\lambda_{i,t} p_t = \lambda_{i,t+1} R_{t+1}$ for every $t \geq T$. By consequence, $\gamma_{i,t} = \gamma_t$ for any $t \geq T + 1$. This implies that $Q_t/Q_{i,t}$ is uniformly bounded from above. According to Proposition 7, there is no bubble. □

Proof of Lemma 6. According to (28), we get $\sum_{t=0}^{\infty} Q_t d_t < \infty$. However we have $d_t > F_i'(L) > \min_i F_i'(L) > 0$ for every t. Therefore, we obtain $\sum_{t=0}^{\infty} Q_t < \infty$. Since $\sup_{i,t} e_{i,t} < \infty$ and $F_i(l_{i,t}) \leq F_i(L)$ for every i, t, we obtain that $\sum_{t=0}^{\infty} Q_t Y_t < \infty$. □

Proof of Lemma 7. We will claim that $\sup_{i,t} Q_t a_{i,t+1} < \infty$. Indeed, (4) is rewritten as

$$Q_{t+1}\frac{R_{t+1}}{p_{t+1}}a_{i,t+1} + f_i Q_{t+1}\Big(\frac{q_{t+1}}{p_{t+1}}l_{i,t+1} + F_i(l_{i,t+1})\Big) \geq 0. \quad (82)$$

Since $Q_t = \frac{R_{t+1}}{p_{t+1}}Q_{t+1}$, (4) is equivalent to

$$Q_t a_{i,t+1} \geq -f_i Q_{t+1}\Big(\frac{q_{t+1}}{p_{t+1}}l_{i,t+1} + F_i(l_{i,t+1})\Big). \quad (83)$$

It is easy to see that $0 \leq Q_t\frac{q_t}{p_t}l_{i,t+1} \leq \frac{q_0}{p_0}L < \infty$. Therefore, we have

$$f_i Q_{t+1}\Big(\frac{q_{t+1}}{p_{t+1}}l_{i,t+1} + F_i(l_{i,t+1})\Big) \leq f_i\frac{q_0}{p_0}L + f_i Q_{t+1}F_i(L). \quad (84)$$

By consequence, we obtain

$$Q_t a_{i,t+1} \geq -f_i\frac{q_0}{p_0}L - f_i Q_{t+1}F_i(L). \quad (85)$$

According to the proof of Lemma 6 , we see that $\lim_{t\to\infty} Q_t = 0$, and hence we get that $\inf_{i,t} Q_t a_{i,t+1} > -\infty$. Since $\sum_{i=1}^m Q_t a_{i,t+1} = 0$, we have $-\infty < \inf_{i,t} Q_t a_{i,t+1} \leq \sup_{i,t} Q_t a_{i,t+1} < \infty$. □

Proof of Lemma 8. We rewrite the budget constraint of agent i at date t as follows

$$Q_t c_{i,t} + Q_t\frac{q_t}{p_t}l_{i,t+1} + Q_t a_{i,t+1} = Q_t(e_{i,t} + F_i(l_{i,t})) + Q_t\frac{q_t}{p_t}l_{i,t} + Q_t\frac{R_t}{p_t}a_{i,t}. \quad (86)$$

According to (18) and (22), we get

$$Q_t\frac{q_t}{p_t} = Q_{t+1}\big(\frac{q_{t+1}}{p_{t+1}} + d_{t+1}\big), \quad Q_t = \frac{R_{t+1}}{p_{t+1}}Q_{t+1}.$$

Therefore, we have

$$\sum_{t=0}^T Q_t c_{i,t} + \sum_{t=1}^T Q_t d_t l_{i,t} + Q_T\big(\frac{q_T}{p_T}l_{i,T+1} + a_{i,T+1}\big) = \sum_{t=0}^T Q_t(e_{i,t} + F_i(l_{i,t})) + \frac{q_0}{p_0}l_{i,0} + \frac{R_0}{p_0}a_{i,0}.$$

By combining this with Lemmas 6 and 7, we obtain that

$$\sup_{T\to\infty}\Big(\sum_{t=0}^T Q_t c_{i,t} + \sum_{t=1}^T Q_t d_t l_{i,t}\Big) < \infty.$$

This implies that there exists the sum $\sum_{t=0}^{\infty}(Q_t c_{i,t} + Q_t d_t l_{i,t})$, and so does $\lim_{t\to\infty} Q_t\big(\frac{q_t}{p_t}l_{i,t+1} + a_{i,t+1}\big)$.

Note that $\lim_{t\to\infty} Q_t c_{i,t} = \lim_{t\to\infty} Q_t(e_{i,t} + F_i(l_{i,t})) = 0$. Then, by using (86), we get (29). □

Proof of Lemma 9. If $\lim_{t\to\infty} Q_t\big(a_{i,t+1}+\frac{q_t}{p_t}l_{i,t+1}\big) > 0$, there exists $T_1 \geq T$ such that $Q_t\big(a_{i,t+1}+\frac{q_t}{p_t}l_{i,t+1}\big) > 0$ for every $t \geq T_1$. Hence, we get

$$\begin{aligned}
&Q_{t+1}\frac{R_{t+1}}{p_{t+1}}a_{i,t+1} + f_iQ_{t+1}\big[\frac{q_{t+1}}{p_{t+1}}l_{i,t+1} + F_i(l_{i,t+1})\big]\\
&\geq Q_{t+1}\frac{R_{t+1}}{p_{t+1}}a_{i,t+1} + Q_{t+1}\big[\frac{q_{t+1}}{p_{t+1}} + d_{t+1}\big]l_{i,t+1} = Q_{t+1}\frac{R_{t+1}}{p_{t+1}}a_{i,t+1} + Q_t\frac{q_t}{p_t}l_{i,t+1} > 0
\end{aligned}$$

for every $t \geq T_1$. This implies that $\mu_{i,t+1} = 0$ for every $t \geq T_1$

Therefore, $\lambda_{i,t}p_t = \lambda_{i,t+1}R_{t+1}$ for every $t \geq T_1$. By consequence, there exists a constant $C_i > 0$ such that $Q_t = C_i\lambda_{i,t}p_t$ for every $t \geq T_1$. According to transversality condition (16), we get $\lim_{t\to\infty} Q_t\big(a_{i,t+1} + \frac{q_t}{p_t}l_{i,t+1}\big) = 0$, contradiction. □

Proof of Propostion 8. If $l_{i,t} = 0$, then condition (30) is satisfied.

If $l_{i,t} > 0$, by combining with $f_i = 1$ and using Lemma 4, we have $d_t = F_i'(l_{i,t}) \leq \frac{F_i(l_{i,t})}{l_{i,t}}$. Therefore, condition (30) is satisfied. By consequence, we have $\lim_{t\to\infty} Q_t\big(a_{i,t+1} + \frac{q_t}{p_t}l_{i,t+1}\big) \leq 0$ for any i. By summing this inequality over i, we obtain $\lim_{t\to\infty} Q_t\frac{q_t}{p_t}L \leq 0$, which implies that bubbles are ruled out. □

Proof of Proposition 10. 1. Since $Q_t \geq Q_{i,t}$, it is easy to see that $FV_0 \leq FV_i$ for any i, and if i-land bubbles exist for some agent i then land bubbles exist.

2. Assume that i-bubble exists for any t, we have $\lim_{t\to\infty} Q_{i,t}\frac{q_t}{p_t} > 0$ for any i. Therefore, we get $\lim_{t\to\infty} Q_t\frac{q_t}{p_t} > 0$. Since both these two limits are finite (less than q_0/p_0) and strictly positive, we see that $\lim_{t\to\infty} Q_t/Q_{i,t} \in (0,\infty)$ for any i. According to Proposition 7 we have $\lim_{t\to\infty} Q_t\frac{q_t}{p_t} = 0$, contradiction.

3. We now assume that $FV_0 = FV_i$ for any i, which implies that $\lim_{t\to\infty} Q_t\frac{q_t}{p_t} = \lim_{t\to\infty} Q_{i,t}\frac{q_t}{p_t}$. If land bubbles exist, we have $\lim_{t\to\infty} Q_t\frac{q_t}{p_t} = \lim_{t\to\infty} Q_{i,t}\frac{q_t}{p_t} \in (0, q_0/p_0)$. Thus, we obtain $\lim_{t\to\infty} Q_{i,t}/Q_t = 1$. According to Proposition 7 we have $\lim_{t\to\infty} Q_t\frac{q_t}{p_t} = 0 = \lim_{t\to\infty} Q_{i,t}\frac{q_t}{p_t}$, contradiction.

□

10 Appendix: Proofs for Section 5

First, we give sufficient conditions for a sequence $\big(p_t, q_t, (c_{i,t}, l_{i,t+1})_{i\in I}\big)_t$ to be an equilibrium. The utility function may satisfy $u_i(0) = -\infty$.

Lemma 10. *If a sequence $\big((c_{i,t}, l_{i,t+1}, \mu_{i,t})_{i\in I}, p_t, q_t\big)_t$ satisfies the following conditions*

(i) $\forall t,\ \forall i,\ c_{i,t} > 0, l_{i,t+1} \geqslant 0,\ \mu_{i,t} \geqslant 0$. $\forall t, p_t = 1, q_t > 0$,

(ii) first order conditions:

$$q_t = \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}\big(q_{t+1} + F_{i,t}'(l_{i,t+1})\big) + \mu_{i,t+1} \tag{87}$$

$$\mu_{i,t+1}l_{i,t+1} = 0, \tag{88}$$

(iii) transversality conditions: $\lim_{t\to\infty} \beta_i^t u_i'(c_{i,t}) q_t l_{i,t+1} = 0$ *for any* i,

(iv) $c_{i,t} + q_t l_{i,t+1} = e_{i,t} + q_t l_{i,t} + F_{i,t}(l_{i,t})$,

(vi) $\sum_{i\in I} l_{i,t} = L$,

then the sequence $\big(p_t, q_t, (c_{i,t}, l_{i,t+1})_{i\in I}\big)_t$ *is an equilibrium for the economy without financial market.*

Proof. Using the same argument in the proof of Proposition 2. □

Check of the example in Section 5.1.1. We now check all conditions in Lemma 10. It is easy to see that the market clearing conditions are satisfied.

Let us check FOCs:

$$q_{2t} = \frac{\beta_B u_B'(c_{B,2t+1})}{u_B'(c_{B,2t})}\big(q_{2t+1} + B_{2t+1}\big) \geq \frac{\beta_A u_A'(c_{A,2t+1})}{u_A'(c_{A,2t})}(q_{2t+1} + A_{2t+1}) \tag{89}$$

$$q_{2t-1} = \frac{\beta_A u_A'(c_{A,2t})}{u_A'(c_{A,2t-1})}(q_{2t} + A_{2t}) \geq \frac{\beta_B u_B'(c_{B,2t})}{u_B'(c_{B,2t-1})}(q_{2t} + B_{2t}). \tag{90}$$

The equality in (89) is satisfied since

$$\frac{\beta_B u_B'(c_{B,2t+1})}{u_B'(c_{B,2t})}\big(q_{2t+1} + B_{2t+1}\big) = \frac{\beta_B(e_{B,2t} - q_{2t})}{q_{2t+1} + B_{2t+1}}\big(q_{2t+1} + B_{2t+1}\big) \tag{91}$$

$$= \beta_B(e_{B,2t} - q_{2t}) = q_{2t}. \tag{92}$$

We now prove the inequality in (89). We have

$$\frac{\beta_A u_A'(c_{A,2t+1})}{u_A'(c_{A,2t})}(q_{2t+1} + A_{2t+1}) = \frac{\beta_A(q_{2t} + A_{2t})}{e_{A,2t+1} - q_{2t+1}}(q_{2t+1} + A_{2t+1}) \tag{93}$$

$$= \frac{\beta_A(\frac{\beta_B}{1+\beta_B}e_{B,2t} + A_{2t})}{\frac{1}{1+\beta_A}e_{A,2t+1}}(\frac{\beta_A}{1+\beta_A}e_{A,2t+1} + A_{2t+1}) \tag{94}$$

By consequence, the inequality in (89) is equivalent to

$$\beta_A(\frac{\beta_B e_{B,2t}}{1+\beta_B} + A_{2t})(\frac{\beta_A e_{A,2t+1}}{1+\beta_A} + A_{2t+1}) \leq \beta_B \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t+1}}{1+\beta_A} \tag{95}$$

which is the condition (37).

We have

$$\frac{\beta_B u_B'(c_{B,2t})}{u_B'(c_{B,2t-1})}(q_{2t} + B_{2t}) = \frac{\beta_B(q_{2t-1} + B_{2t-1})}{e_{B,2t} - q_{2t}}(q_{2t} + B_{2t}) \tag{96}$$

$$= \frac{\beta_B(\frac{\beta_A}{1+\beta_A}e_{A,2t-1} + B_{2t-1})}{\frac{1}{1+\beta_B}e_{B,2t}}(\frac{\beta_B}{1+\beta_B}e_{B,2t} + B_{2t}) \tag{97}$$

By consequence, the inequality in (90) is equivalent to

$$\beta_B(\frac{\beta_A e_{A,2t-1}}{1+\beta_A} + B_{2t-1})(\frac{\beta_B e_{B,2t}}{1+\beta} + B_{2t}) \leq \beta_A \frac{e_{B,2t}}{1+\beta_B}\frac{e_{A,2t-1}}{1+\beta_A} \tag{98}$$

which is the condition (38).

We now check TVCs. We have

$$\beta_A^{2t} u_A'(c_{A,2t}) q_{2t} l_{A,2t+1} = 0 \tag{99}$$

$$\beta_A^{2t-1} u_A'(c_{A,2t-1}) q_{2t-1} l_{A,2t} = \frac{\beta_A^{2t-1}}{c_{A,2t-1}} q_{2t-1} = \beta_A^{2t} \to 0. \tag{100}$$

Similarly, we also have

$$\beta_B^{2t} u_B'(c_{B,2t}) q_{2t} l_{B,2t+1} = \beta_B^{2t+1} \to 0 \tag{101}$$

$$\beta_B^{2t-1} u_B'(c_{B,2t-1}) q_{2t-1} l_{B,2t} = 0. \tag{102}$$

We finally verify that, for each $t \geq 0$,

$$\frac{\beta_B u_B'(c_{B,2t+1})}{u_B'(c_{B,2t})} \geq \frac{\beta_A u_A'(c_{A,2t+1})}{u_A'(c_{A,2t})} \tag{103}$$

$$\frac{\beta_B u_B'(c_{B,2t})}{u_B'(c_{B,2t-1})} \leq \frac{\beta_A u_A'(c_{A,2t})}{u_A'(c_{A,2t-1})}. \tag{104}$$

Indeed, condition (103) is rewritten as

$$\frac{\beta_B(e_{B,2t} - q_{2t})}{q_{2t+1} + B_{2t+1}} \geq \frac{\beta_A(\frac{\beta_B}{1+\beta_B} e_{B,2t} + A_{2t})}{\frac{1}{1+\beta_A} e_{A,2t+1}}. \tag{105}$$

Since $q_{2t} = \frac{\beta_B}{1+\beta_B} e_{B,2t}, q_{2t+1} = \frac{\beta_A}{1+\beta_A} e_{A,2t+1}$, condition (103) is equivalent to

$$\beta_A\left(\frac{\beta_B e_{B,2t}}{1+\beta_B} + A_{2t}\right)\left(\frac{\beta_A e_{A,2t+1}}{1+\beta_A} + B_{2t+1}\right) \leq \beta_B \frac{e_{B,2t}}{1+\beta_B} \frac{e_{A,2t+1}}{1+\beta_A}. \tag{106}$$

This is condition (39).

By the same argument, we see that condition (104) is equivalent to

$$\beta_B\left(\frac{\beta_A e_{A,2t-1}}{1+\beta_A} + B_{2t-1}\right)\left(\frac{\beta_B e_{B,2t}}{1+\beta} + A_{2t}\right) \leq \beta_A \frac{e_{B,2t}}{1+\beta_B} \frac{e_{A,2t-1}}{1+\beta_A}. \tag{107}$$

This is condition (40).

□

11 Appendix: Existence of equilibrium for the intermediate economy $\tilde{\mathcal{E}}$

In this appendix, we present a proof of the existence of equilibrium for the economy $\tilde{\mathcal{E}}$. We allow for non-stationary technologies, i.e., the production functions $F_{i,t}$ depend on both i and t.

11.1 Existence of equilibrium for truncated economies

For each $T \geq 1$, we define $T-$ truncated economy $\tilde{\mathcal{E}}^T$ as $\tilde{\mathcal{E}}$ but there are no activities from period $T+1$ to the infinity, i.e., $c_{i,t} = l_{i,t} = b_{i,t} = 0$ for every $i = 1, \ldots, m$, $t \geq T+1$.

We then define the bounded economy $\tilde{\mathcal{E}}_b^T$ as $\tilde{\mathcal{E}}^T$ but consumption level $(c_{i,t})_{t=0}^T$, land holding $(l_{i,t})_{t=1}^T$, and asset holding $(b_{i,t})_{t=1}^T$ are respectively bounded in the following sets:

$$\mathcal{C} \;:=\; [0, B_c]^{T+1}, \quad \mathcal{L} := [0, B_l]^T, \quad \mathcal{A} := \prod_{t=1}^{T}[-B_b, B_b]^T,$$

where $B_c > \max_{t \le T} \sum_{i=1}^{m} \Big(e_{i,t} + F_{i,t}(B_l)\Big)$, $B_l > L$, and $B_b = m(B_c + B_l)$.

The economy $\tilde{\mathcal{E}}_b^T$ depends on bounds B_c, B_l, B_b. We write $\tilde{\mathcal{E}}_b^T(B_c, B_l, B_b)$.

Let us denote

$$\mathcal{X}_b := \mathcal{C} \times \mathcal{L} \times \mathcal{A}, \quad \mathcal{X} := (\mathcal{X}_b)^m \tag{108}$$

We then define

$$\begin{aligned} \mathcal{P} &:= \{z_0 = (p, q, r) : r_0 = 0, q_T = 0; & (109)\\ &\quad p_t, q_t, r_t \ge 0; 2p_t + q_t + r_t = 1 \quad \forall t = 0, \dots, T\} & (110)\\ \Phi &:= \mathcal{P} \times \mathcal{X}. & (111)\end{aligned}$$

An element $z \in \Phi$ is in the form $z = (z_i)_{i=0}^m$ where $z_0 := (p, q, r)$, $z_i := (c_i, l_i, b_i)$ for each $i = 1, \dots, m$.

The following remark is to ensure that the asset volume $(b_{i,t})$ is bounded.

Remark 16. *If $z \in \Phi$ is an equilibrium for the economy $\tilde{\mathcal{E}}$ then $c_{i,t} \in [0, B_c)$, $l_{i,t} \in [0, L]$. By using the fact that $2p_t + q_t + r_t = 1$, we get that $b_{i,t} \le B_c + B_l$ for any i, t. Since $\sum_{i=1}^{m} b_{i,t} = 0$, we obtain that $b_{i,t} \in [-B_b, B_b]$ for any i and any t.*

Proposition 11. *Under Assumptions (1-4), there exists an equilibrium $(p, q, r, (c_i, l_i, b_i)_{i=1}^m)$, with $2p_t + q_t + r_t = 1$, for the economy $\tilde{\mathcal{E}}_b^T(B_c, B_l, B_b)$.*

Proof. We first define

$$\begin{aligned} C_i^T(p, q, r) := \big\{&(c_{i,t}, l_{i,t+1}, b_{i,t+1})_{t=0}^T \in \mathcal{X} : \text{(a) } l_{i,T+1} = b_{i,T+1} = 0,\\ &\text{(b) } p_0 c_{i,0} + q_0 l_{i,1} + b_{i,1} \le p_0 e_{i,0} + p_0 F_{i,0}(l_{i,0}) + q_0 l_{i,0}\\ &\text{(c) for each } 1 \le t \le T:\\ &\quad 0 \le r_t b_{i,t} + f_i\Big(q_t l_{i,t} + p_t F_{i,t}(l_{i,t})\Big)\\ &\quad p_t c_{i,t} + q_t l_{i,t+1} + b_{i,t+1} \le p_t e_{i,t} + p_t F_{i,t}(l_{i,t}) + q_t l_{i,t} + r_t b_{i,t}\big\}. \end{aligned}$$

We also define $B_i^T(p, q, r)$ as follows.

$$\begin{aligned} B_i^T(p, q, r) := \big\{&(c_{i,t}, l_{i,t+1}, b_{i,t+1})_{t=0}^T \in \mathcal{X} : \text{(a) } l_{i,T+1} = b_{i,T+1} = 0,\\ &\text{(b) } p_0 c_{i,0} + q_0 l_{i,1} + b_{i,1} < p_0 e_{i,0} + p_0 F_{i,0}(l_{i,0}) + q_0 l_{i,0}\\ &\text{(c) for each } 1 \le t \le T:\\ &\quad 0 < r_t b_{i,t} + f_i\Big(q_t l_{i,t} + p_t F_{i,t}(l_{i,t})\Big)\\ &\quad p_t c_{i,t} + q_t l_{i,t+1} + b_{i,t+1} < p_t e_{i,t} + p_t F_{i,t}(l_{i,t}) + q_t l_{i,t} + r_t b_{i,t}\big\}. \end{aligned}$$

Lemma 11. *$B_i^T(p, q, r) \ne \emptyset$ and $\bar{B}_i^T(p, q, r) = C_i^T(p, q, r)$.*

Proof. It can be easily proved by using the following remark: since $e_{i,0}, l_{i,0} > 0$ and $(p_0, q_0) \neq (0,0)$, we always have $p_0 e_{i,0} + p_0 F_{i,0}(l_{i,0}) + q_0 l_{i,0} > 0$. □

Lemma 12. *$B_i^T(p,q,r)$ is lower semi-continuous correspondence on $\mathcal{P}$. $C_i^T(p,q,r)$ is continuous on $\mathcal{P}$ with compact convex values.*

Proof. It is clear since $B_i^T(p,q,r)$ is nonempty and has open graph. □

We now define correspondences.

First, we define φ_0 (for additional agent 0) : $\mathcal{X} \to 2^{\mathcal{P}}$ by

$$\varphi_0((z_i)_{i=1}^m) := \arg\max_{(p,q,r)\in\mathcal{P}} \Big\{ \sum_{t=0}^{T} \Big[p_t \sum_{i=1}^{m} \big(c_{i,t} - e_{i,t} - F_{i,t}(l_{i,t}) \big) \Big] + \sum_{t=0}^{T-1} \Big[q_t \sum_{i=1}^{m} (l_{i,t+1} - l_{i,t}) \Big] + \sum_{t=1}^{T} \Big[r_t (-\sum_{i=1}^{m} b_{i,t}) \Big] \Big\}.$$

For each $i = 1, \ldots, m$, we define $\varphi_i : \mathcal{P} \to 2^{\mathcal{X}}$

$$\varphi_i((p,q,r)) \quad := \quad \arg\max_{(c_i,l_i,b_i)\in C_i(p,q,r)} \Big\{ \sum_{t=0}^{T} \beta_i^t u_i(c_{i,t}) \Big\}.$$

Lemma 13. *The correspondence φ_i is upper semi-continuous and non-empty, convex, compact valued for each $i = 0, 1, \ldots, m+1$.*

Proof. This is a direct consequence of the Maximum Theorem. □

According to the Kakutani Theorem, there exists $(\bar{p}, \bar{q}, \bar{r}, (\bar{c}_i, \bar{l}_i, \bar{b}_i)_{i=1}^m)$ such that

$$(\bar{p}, \bar{q}, \bar{r}) \in \varphi_0((\bar{c}_i, \bar{l}_i, \bar{b}_i)_{i=1}^m) \tag{112}$$
$$(\bar{c}_i, \bar{l}_i, \bar{b}_i) \in \varphi_i((\bar{p}, \bar{q}, \bar{r})). \tag{113}$$

Denote, for each $t \geq 1$,

$$\bar{X}_t := \sum_{i=1}^{m} (\bar{c}_{i,t} - e_{i,t} - F_{i,t}(\bar{l}_{i,t})), \quad \bar{Y}_t := \sum_{i=1}^{m} (\bar{l}_{i,t+1} - \bar{l}_{i,t}), \quad \bar{Z}_t := -\sum_{i=1}^{m} \bar{b}_{i,t}.$$

For every $(p,q,r) \in \mathcal{P}$, we have

$$\sum_{t=0}^{T} (p_t - \bar{p}_t)\bar{X}_t + \sum_{t=0}^{T-1} (q_t - \bar{q}_t)\bar{Y}_t + \sum_{t=1}^{T} (r_t - \bar{r}_t)\bar{Z}_t \leq 0. \tag{114}$$

By summing budget constraints at date T over i, we get that

$$\bar{p}_T \bar{X}_T + \bar{r}_T \bar{Z}_T \leq 0.$$

By consequence, we have, for every $(p_T, r_T) \geq 0$ with $2p_T + r_T = 1$,

$$p_T \bar{X}_T + r_t \bar{Z}_t \leq \bar{p}_T \bar{X}_T + \bar{r}_t \bar{Z}_t \leq 0.$$

Therefore, we have $\bar{X}_T, \bar{Z}_T \leq 0$ for any t, which means that

$$\sum_{i=1}^{m} \bar{c}_{i,T} \leq \sum_{i=1}^{m} (e_{i,T} + F_{i,t}(\bar{l}_{i,T})) \text{ and } -\sum_{i=1}^{m} \bar{b}_{i,T} \leq 0. \tag{115}$$

By summing the budget constraints over i at date t, we obtain that

$$\bar{p}_t\bar{X}_t + \bar{q}_t\bar{Y}_t + \bar{r}_t\bar{Z}_t \le \bar{Z}_{t+1}.$$

Since $\bar{Z}_T \le 0$, we get $\bar{p}_{T-1}\bar{X}_{T-1} + \bar{q}_{T-1}\bar{Y}_{T-1} + \bar{r}_{T-1}\bar{Z}_{T-1} \le 0$. By consequence, we have that $p_{T-1}\bar{X}_{T-1}+p_{T-1}\bar{Y}_{T-1}+r_{T-1}\bar{Z}_{T-1} \le 0$ for any t. This implies that $\bar{X}_{T-1}, \bar{Y}_{T-1}, \bar{Z}_{T-1} \le 0$. Repeating this argument, we obtain that $\bar{X}_t, \bar{Y}_t, \bar{Z}_t \le 0$ for any t, which means that

$$\sum_{i=1}^{m} \bar{c}_{i,t} \le \sum_{i=1}^{m}(e_{i,t} + F_{i,t}(\bar{l}_{i,t}))$$

$$\sum_{i=1}^{m}(\bar{l}_{i,t+1} - \bar{l}_{i,t}) \le 0$$

$$-\sum_{i=1}^{m} \bar{b}_{i,t} \le 0.$$

Lemma 14. *$\bar{p}_t, \bar{q}_t, \bar{r}_t > 0$ for $t = 0, \ldots, T$.*

Proof. We see that $\sum\limits_{i=1}^{m} \bar{l}_{i,t} \le L$ and $\sum\limits_{i=1}^{m} \bar{b}_{i,t} \ge 0$. Hence, we have

$$\sum_{i=1}^{m} \bar{c}_{i,t} \le \sum_{i=1}^{m}(e_{i,t} + F_{i,t}(L))$$

which allows us to prove that $\bar{p}_t > 0$. Indeed, if $\bar{p}_t = 0$ then $c_{i,t} = B_c > \sum\limits_{i=1}^{m}(e_{i,t} + F_{i,t}(L))$, a contradiction.

If $\bar{q}_t = 0$, then $\bar{l}_{i,t} = B_l > L$ for any i, contradiction.

If $\bar{r}_t = 0$ then $\bar{b}_{i,t} = -B_a$ for any i, which implies that $\sum\limits_{i=1}^{m} \bar{b}_{i,t} < 0$, contradiction. □

Lemma 15. *$\bar{X}_t = \bar{Z}_t = \bar{Y}_t = 0$.*

Proof. Using $\bar{p}_t\bar{X}_t + \bar{q}_t\bar{Y}_t + \bar{r}_t\bar{Z}_t \le 0$. □

Lemma 16. *The optimality of (c_i, l_i, b_i).*

Proof. It is clear since $(\bar{c}_i, \bar{l}_i, \bar{b}_i) \in \varphi_i((\bar{p}, \bar{q}, \bar{r}))$. □

We have just proved that $(\bar{p}, \bar{q}, \bar{r}, (\bar{c}_i, \bar{l}_i, \bar{b}_i)_{i=1}^m)$ is an equilibrium for the economy $\tilde{\mathcal{E}}_b^T$. □

We see that any equilibrium of the economy $\tilde{\mathcal{E}}^T$, if it exists, we can normalize by setting $2p_t + q_t + r_t = 1$. We also observe that consumption, land and asset allocations of such an equilibrium are belong to $\mathcal{X}$, defined in (108). Hence, we obtain that

Proposition 12. *An equilibrium $(\bar{p}, \bar{q}, \bar{r}, (\bar{c}_i, \bar{l}_i, \bar{b}_i)_{i=1}^m)$, with $2p_t + q_t + r_t = 1$, of $\tilde{\mathcal{E}}_b^T$ is an equilibrium for $\tilde{\mathcal{E}}^T$.*

11.2 Existence of equilibrium for the infinite-horizon economy

Let us denote $W_t := \sum_{i=1}^{m} \big(e_{i,t} + F_{i,t}(L)\big)$. We need the following assumption to ensure that the utility of every agent is finite.

Assumption 6. *$\sum_{t=0}^{\infty} \beta_i^t u_i(W_t) < \infty$ for each i.*

Remark 17. *For simplicity of notation, in what follows, we write F_i instead of $F_{i,t}$ since our argument still holds for non-stationary technologies.*

Proposition 13. *Under Assumptions (1-4) and 7, there exists an equilibrium for the economy $\tilde{\mathcal{E}}$.*

Proof. We present a proof in the spirit Le Van and Pham (2015).

We have shown that there exists an equilibrium, say $\big(\bar{p}^T, \bar{q}^T, \bar{r}^T, (\bar{c}_i^T, \bar{l}_i^T, \bar{b}_i^T)_i\big)$, for each truncated economy $\tilde{\mathcal{E}}^T$. Recall that $2\bar{p}_t^T + \bar{q}_t^T + \bar{r}_t^T = 1$.

It is clear that $0 < \bar{c}_{i,t}^T < W_t$ for each $t \geq 0$, and $\bar{l}_{i,t}^T \in [0, L]$, $\bar{p}_t^T, \bar{p}_t^T, \bar{r}_t^T \in [0,1]$ and $2\bar{p}_t^T + \bar{q}_t^T + \bar{r}_t^T = 1$.

We define a sequence (B_t) as by $B_1 = W_1, B_{t+1} = B_t + W_{t+1} \quad \forall t$. It is easy to prove that $\bar{b}_{i,t}^T < B_t$ for any t and any T. Since $\sum_{i=1}^{m} \bar{b}_{i,t}^T = 0$ we have $\bar{b}_{i,t}^T \in (-mB_t, mB_t)$ for any t and any T. Therefore, without loss of generality, we can assume that

$$\big(\bar{p}^T, \bar{q}^T, \bar{r}^T, (\bar{c}_i^T, \bar{l}_i^T, \bar{b}_i^T)_i)\big) \xrightarrow{T\to\infty} \big(\bar{p}, \bar{q}, \bar{r}, (\bar{c}_i, \bar{l}_i, \bar{b}_i)_i\big)$$

for the product topology.

We will prove that $\big(\bar{p}, \bar{q}, \bar{r}, (\bar{c}_i, \bar{l}_i, \bar{b}_i)_i\big)$ is an equilibrium for the economy $\tilde{\mathcal{E}}$. The market clearing conditions are trivial. We will prove that all prices are strictly positive and the allocation $(\bar{c}_i, \bar{l}_i, \bar{b}_i)$ is optimal.

Let (c_i, l_i, b_i) be a feasible allocation of the problem $P_i(\bar{p}, \bar{q}, \bar{r})$. First, we prove that $\sum_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.

We define $(c'_{i,t}, l'_{i,t+1}, b'_{i,t+1})_{t=0}^T$ as follows:

$$\begin{aligned}(c'_{i,}, l'_{i,t+1}, b'_{i,t+1}) &= (c_{i,t}, l_{i,t+1}, b_{i,t+1}) \text{ if } t \leq T-1\\ (c'_{i,T}, l'_{i,T+1}, b'_{i,T+1}) &= (e_{i,T}, 0, 0).\end{aligned}$$

We see that $(c'_{i,t}, l'_{i,t+1}, b'_{i,t+1})_{t=0}^T$ belongs to $C_i^T(\bar{p}, \bar{q}, \bar{r})$.[15]

Since $l_{i,0} > 0$ we have $\bar{q}_0 l_{i,0} + \bar{p}_0 F_i(l_{i,0}) > 0$, and hence $B_i^T(\bar{p}, \bar{q}, \bar{r}) \neq \emptyset$, therefore there exists a sequence $\Big((c_{i,t}^n, l_{i,t+1}^n, b_{i,t+1}^n)_{t=0}^T\Big)_{n=0}^{\infty} \in B_i^T(\bar{p}, \bar{q}, \bar{r})$ with $l_{i,T+1}^n = 0$, $b_{i,T+1}^n = 0$, and this sequence converges to $(c'_{i,t}, l'_{i,t+1}, b'_{i,t+1})_{t=0}^T$ when n tends to infinity. We have

$$\begin{aligned}\bar{p}_t c_{i,t}^n + \bar{q}_t l_{i,t+1}^n + b_{i,t+1}^n &< \bar{p}_t e_{i,t} + \bar{q}_t l_{i,t}^n + \bar{p}_t F_i(l_{i,t}^n) + \bar{r}_t b_{i,t}^n\\ f_i\Big(\bar{q}_t l_{i,t}^n + \bar{p}_t F_i(l_{i,t}^n)\Big) + \bar{r}_t b_{i,t}^n &> 0.\end{aligned}$$

We can chose s_0 high enough such that $s_0 > T$ and for every $s \geq s_0$, we have

$$\begin{aligned}\bar{p}_t^s c_{i,t}^n + \bar{q}_t^s l_{i,t+1}^n + b_{i,t+1}^n &< \bar{p}_t^s e_{i,t} + \bar{q}_t^s l_{i,t}^n + \bar{p}_t^s F_i(l_{i,t}^n) + \bar{r}_t^s b_{i,t}^n\\ f_i\Big(\bar{q}_t^s l_{i,t}^n + \bar{p}_t^s F_i(l_{i,t}^n)\Big) + \bar{r}_t^s b_{i,t}^n &> 0.\end{aligned}$$

[15] Thank to borrowing constraints, we can choose $c'_{i,T} = e_{i,T}$.

It means that $(c^n_{i,t}, l^n_{i,t+1}, b^n_{i,t+1})_{t=0}^T \in C^T_i(\bar{p}^s, \bar{q}^s, \bar{r}^s)$. Therefore, we get $\sum\limits_{t=0}^{T} \beta_i^t u_i(c^n_{i,t}) \leq \sum\limits_{t=0}^{s} \beta_i^t u_i(\bar{c}^s_{i,t})$.
Let s tend to infinity, we obtain $\sum\limits_{t=0}^{T} \beta_i^t u_i(c^n_{i,t}) \leq \sum\limits_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$.
Let n tend to infinity, we have $\sum\limits_{t=0}^{T} \beta_i^t u_i(c'_{i,t}) \leq \sum\limits_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$ for every T. As a consequence, we have: for every T

$$\sum_{t=0}^{T-1} \beta_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t}).$$

Let T tend to infinity, we obtain $\sum\limits_{t=0}^{\infty} \beta_i^t u_i(c_{i,t}) \leq \sum\limits_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$. It means that we have proved the optimality of $(\bar{c}_i, \bar{l}_i, \bar{b}_i)$.

Now, we prove that $\bar{p}_t > 0$. Indeed, if $\bar{p}_t = 0$, the agent i can freely improve her consumption to obtain a level of utility, which is higher than $\sum\limits_{t=0}^{\infty} \beta_i^t u_i(\bar{c}_{i,t})$. This contradicts the optimality of $(\bar{c}_i, \bar{l}_i, \bar{b}_i)$.

We have $\bar{q}_t$ is strictly positive because $\bar{p}_t > 0$ and the utility function of agent i is strictly increasing. $\bar{r}_t > 0$ is implied by $\bar{p}_t > 0$.

□

References

Araujo A., Novinski R., Pascoa M.R., *General equilibrium, wariness and efficient bubbles*, Journal of Economic Theory 46, p. 785-811 (2011).

Araujo A., Pascoa M.R., Torres-Martinez J.P., *Long-lived collateralized assets and bubbles*, Journal of Mathematical Economics 47, p. 260-271 (2011).

Becker R., S. Bosi, C. Le Van and T. Seegmuller, *On existence and bubbles of Ramsey equilibrium with borrowing constraints*, Economic Theory 58, 329-353 (2015).

Becker R. and T. Mitra, *Efficient Ramsey equilibria*, Macroeconomic Dynamics 16, 18-32 (2012).

Bewley T., *The optimal quantity of money*, In: Kareken J., Wallace N. (eds.) Models of Monetary Economics, Minneapolis: Federal Reserve Bank (1980).

Bosi S., C. Le Van and N-S. Pham, *Asset bubbles and efficiency in a generalized two-sector model*, EPEE working paper (2015).

Brunnermeier M.K. and M. Oehmke, *Bubbles, financial crises, and systemic risk*, Handbook of the Economics of Finance, vol. 2 (2012).

Farhi E., Tirole J., *Bubbly liquidity*, Review of Economic Studies 79, 678-706 (2012).

Geanakoplos J., Zame W., *Collateral and the enforcement of intertemporal contracts*. Working paper (2002).

Huang K. X. D. and Werner J., *Asset price bubbles in Arrow-Debreu and sequential equilibrium*, Economic Theory 15, 253-278 (2000).

Kamihigashi T., *A simple proof of the necessity of the transversality condition*, Economic Theory 20, 427-433 (2002).

Kiyotaki N. and Moore J., *Credit cycles*. Journal of Political Economy, 105(2), p. 211-248, 1997.

Kocherlakota, N. R., *Bubbles and constraints on debt accumulation*, Journal of Economic Theory, 57, 245 - 256 (1992).

Kocherlakota, N. R., *Injecting rational bubbles*, Journal of Economic Theory, 142, 218 - 232 (2008).

Le Van C. and Pham N.S., *Financial asset bubble with heterogeneous agents and endogenous borrowing constraints*, Working paper, 2014.

Le Van C., and Pham N.S., *Intertemporal equilibrium with financial asset and physical capital*, Economic Theory (forthcoming 2015).

Liu Z., Wang P. and Zha, T. *Land-price dynamics and macroeconomic fluctuations*, Econometrica 82, 1147-1184 (2013).

Lucas R. E., *Asset prices in an exchange economy*, Econometrica 46, pp. 1429-1445 (1978).

Miao J., *Introduction to economic theory of bubbles*, Journal of Mathematical Economics 53, 130-136 (2014).

Miao J. and P. Wang, *Bubbles and total factor productivity*, American Economic Review: Papers and Proceedings 102, 82-87 (2012)

Miao J. and P. Wang, *Bubbles and credit constraints*, working paper, Boston University (2015).

Montrucchio, L., *Cass transversality condition and sequential asset bubbles*, Economic Theory, 24, 645-663 (2004).

Pascoa M.R., Petrassi M., Torres-Martinez J.P., *Fiat money and the value of binding portfolio constraints*, Economic Theory 46, p. 189-209 (2011).

Santos M. S., and Woodford M. *Rational asset pricing bubbles*, Econometrica, 65, 19-57 (1997).

Tirole J., *On the possibility of speculation under rational expectations*, Econometrica 50, 1163-1181 (1982).

Tirole J., *Asset bubbles and overlapping generations*, Econometrica 53, 1499-1528 (1985).

Weil F., *On the possibility of price decreasing bubbles*, Econometrica, 58, 1467-1474 (1990).

Werner J., *Speculative trade under ambiguity*, mimeo (2014).