Financial bubbles and capital accumulation in altruistic economies*

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Abstract

We consider an overlapping generations model à la Diamond (1965) with altruistic parents and an asset (or land) bringing non-stationary positive dividends (or fruits). We study the global dynamics of capital stocks and asset values. Asset price bubbles are also investigated.

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1 Introduction

According to the literature on pure rational bubbles[1] à la Tirole (1985), a bubble may coexist with physical capital because (1) agents want to buy the asset at any date (the young buy the bubble from the old) and (2) the real interest rate of the economy without financial asset is lower than the population growth rate (the economy experiences capital overaccumulation).[2] Although this literature is huge, very few papers have tackled the issue of bubble existence when dividends are positive. Many unaddressed questions on bubbles with positive dividend remain. Why do these bubbles arise? What are their dynamic properties? How do the capital and asset values interfere over time? What is the difference between bubbles of assets with and without dividends?

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[1] A bubble is pure when the asset yields no dividends.
[2] The reader is referred to Miao (2014) for an introduction to bubbles in infinite-horizon models and to Brunnermeier and Oehmke (2013) for a survey on bubbles in OLG models with asymmetric information or heterogeneous beliefs.

Our goal is to address these open issues. In addition, we generalize Tirole (1985) with a kind of altruism. Altruism matters affecting the offspring's saving of and the portfolio composition. Therefore, the novelty of the paper is twofold and rests on the introduction of forward (or descending) altruism and a financial asset (or land) bringing non-stationary positive dividends (or fruits) in the overlapping generations benchmark à la Diamond (1965).

Standard Inada condition ensures the existence of an interior equilibrium. Indeed, a low productivity entails the equilibrium failure because households prefer to invest in financial asset instead of physical capital.

Results on existence are complemented by a global analysis of equilibrium including the case of bubbly equilibria. As in the standard literature (Tirole, 1982; Kocherlakota, 1992; Santos and Woodford, 1997; Huang and Werner, 2000), we say that a bubble exists at equilibrium if the equilibrium price of financial asset exceeds the present discounted value of its dividends, that is its *fundamental value*. In short, we call a *bubble* the difference between the asset price and the fundamental value. This equals the value at infinity of one unit of asset. In particular, when dividend is zero at any date, the bubble coincides with the asset price and it is said to be *pure*. This kind of asset is considered in Tirole (1985) and called *fiat money* by some authors (Bewley, 1980; Weil, 1987).

First, we prove that if there is no bubbly equilibrium, then the economy has a unique equilibrium, and, of course, this equilibrium is bubbleless. So, the main part of our analysis focuses on multiple equilibria where bubbles may appear.

One of our main results is that a bubble exists only if the sum over time of ratios of dividend to production is finite. As a consequence, in a bounded economy,[3] a bubble exists only if the sum over time of dividends is finite. This entails a number of implications. For instance, there is no bubble at the steady state if dividends are strictly positive. This property holds whatever the level of interest rate. By contrast, as proved by Tirole (1985), a pure bubble may arise at the steady state: this is the very difference between bubbles on assets with and without dividend. Now, we understand why Weil (1990) need to require the dividends to be zero after a finite number of periods.

We also show that, in a bounded economy with high interest rates, bubbles are ruled out. This result is independent of the level of asset dividends and, in this respect, quintessential. Of course, it covers Tirole (1985), where dividends are zero at any date, and rests on the following intuition. As seen above, in a bounded economy, bubbles are excluded when dividends do not converge to zero. However, under high interest rates, even if dividends converge to zero, households prefer to invest in physical capital instead of financial asset. Hence, the asset value tends to zero. If the value of one asset unit is asymptotically zero, there is no bubble.

Summing up, we obtain two necessary conditions for bounded economies, under which bubble may arise: (1) a low interest rate and (2) a finite sum of dividends. Interestingly, we show that along a bubbly equilibrium, capital stocks converge either to the steady state of the economy without financial asset or to the level at which the interest rate equals the rate of population growth.

To illustrate these theoretical findings, we provide some examples of assets with positive dividends giving rise to a bubble. In the case of Cobb-Douglas and linear technologies, we obtain a continuum of bubbly equilibria, a novelty with respect to the existing literature. Closed forms are also computed. We find that a higher degree of forward altruism lowers the interest rate in the economy without financial asset. In this respect, we can say that descendent altruism promotes bubbles.

[3]Output per capita is uniformly bounded from above.

In the last part of the paper, we revisit and bridge more bubble, interest rate and asset price. The seminal article by Tirole (1985) finds out that existence of pure bubbles requires a low interest rate. Such conclusion rests on the boundedness of aggregate output, including asset dividends. In the case of high interest rate, if a bubble exists, the asset values grow to infinity and the equilibrium feasibility is violated. However, we argue that, in the case of unbounded growth, even if the interest rate is high, the growth of asset values satisfies the equilibrium conditions, namely the positivity of capital stock, and bubbles may occur. Moreover, in such an economy, dividends are no longer required to be bounded. This is also an added value of our paper.[4]

At a first sight, we may be convinced that asset prices increase along a bubbly equilibrium. However, we provide a counterexample of bubbly equilibrium along which asset prices may increase, decrease or even fluctuate in time. This means that no causal link exists between bubble existence and monotonicity of asset prices.

The rest of the paper is organized as follows. Section 2 introduces the economic fundamentals. Section 3 presents some equilibrium properties and a formal definition of bubble. Section 5 provides some general results on equilibrium transition for bubbles and capital. Section 6 and Section 7 focus on particular cases and global dynamics. Technical proofs are gathered in Appendices.

2 Model

We consider a two-period OLG model of rational bubbles in the spirit of Diamond (1965), Tirole (1985) and Weil (1987).

Firms. Technology is represented by a constant returns to scale production function $F(K, L)$ where K_t and L_t are the aggregate capital and the labor forces. Profit maximization under complete capital depreciation implies

$$R_t = R(k_t) \equiv f'(k_t) \text{ and } w_t = w(k_t) \equiv f(k_t) - k_t f'(k_t) \quad (1)$$

where $k_t \equiv K_t/L_t$ denotes the capital intensity, $f(k_t) \equiv F(k_t, 1)$, R_t and w_t represent the return on capital and the wage rate.

Generations. Assume that there are N_t new individuals enter the economy at time t. The growth factor of population is supposed to be constant: $n = N_{t+1}/N_t$.

Households. Each young agent lives for two periods and supplies one unit of labor. Assume that preferences of households are rationalized by an additively separable utility function

$$u(c_t, d_{t+1}) \equiv u(c_t) + \beta u(d_{t+1})$$

where β represents the degree of patience, while c_t and d_{t+1} denote the consumption demands at time t and $t+1$ of a household born at time t. We assume that u is concave and strictly increasing.

Agents save through a portfolio (a_t, s_t) of financial asset and physical capital. Consumption prices are normalized to one. q_t and $\delta_t \geq 0$ denote the asset price in consumption units and the dividend, while $b_t \equiv q_t a_t$ and $\xi_t \equiv \delta_t a_t$ the values of asset and dividend respectively. The sequence of dividends (δ_t) is assumed to be exogenous.

Once households buy the asset a_t, they will be able to resell it tomorrow and perceive dividends (in term of consumption good). This asset can also be interpreted as a Lucas' tree or land.

[4]To connect growth and pure bubbles, the reader is referred to Hirano and Yanagawa (2015), Bosi and Pham (2016) and references therein.

Budget constraints of household born at date t is written

$$c_t + s_t + q_t a_t \leq w_t + g_t \tag{2}$$

$$d_{t+1} + n g_{t+1} \leq R_{t+1} s_t + (q_{t+1} + \delta_{t+1}) a_t \tag{3}$$

$$x d_{t+1} \leq n g_{t+1} \tag{4}$$

where g_{t+1} represents the bequests from parents to offspring and x is the degree of forward (or descending) altruism (Michel, Thibault and Vidal, 2006). We introduce this kind of ad hoc altruism to have a tractable model. See Bosi et al. (2015) for bubbles in an OLG model where altruism à la Barro (1974) is introduced through recursive utility.

The market clearing conditions sum up to $N_t s_t = K_{t+1}$ and $N_t a_t = N_{t+1} a_{t+1}$, that is, respectively to

$$s_t = n k_{t+1} \text{ (capital)} \tag{5}$$

$$a_t = n a_{t+1} \text{ (financial asset)} \tag{6}$$

Definition 1. *A positive list $(q_t, R_t, w_t, c_t, d_{t+1}, g_{t+1}, s_t, a_t, k_{t+1})_t$ is an intertemporal equilibrium for the economy with forward altruism if, given $(q_t, q_{t+1}, R_t, w_t, g_t)$: (i) the allocations $(c_t, d_{t+1}, g_{t+1}, s_t, a_t)$ maximize $u(c_t, d_{t+1})$ subject to constraints (2, 3, 4) and (ii) conditions (5, 6) are satisfied.*

Assumption 1. $\bar{\xi} \equiv \sup_t \xi_t < \infty$ *or, equivalently,* $\sup_t (\delta_t a_0 / n^t) < \infty$.

For simplicity, in the rest of the paper we focus on equilibra with $k_t > 0$ for any t. In this case, the consumer's program leads to an (equilibrium) no-arbitrage condition:

$$q_t = \frac{q_{t+1} + \delta_{t+1}}{R_{t+1}} \tag{7}$$

Remark 1. *At equilibrium, budget constraints become binding. Combining them with (5, 6) and (7) we obtain a sequence $(b_t, k_{t+1})_{t \geq 0}$ which is a reduced and equivalent form of equilibrium. Thus, from now on, we will refer to this sequence as an equilibrium.*

3 Equilibrium

This section provides some basic equilibrium properties and introduces the notion of bubble.

Constraints (2), (3) and (4) entail $n g_t = x d_t$. Combining it with the no-arbitrage condition (7), we observe that the household's total saving $s_t + b_t = n k_{t+1} + b_t$ only depends on $w_t + g_t$ and R_{t+1}. So, we can assume that

$$n k_{t+1} + b_t = \mathcal{S}_x (w_t + g_t, R_{t+1}) \tag{8}$$

where $\mathcal{S}_x$ is interpreted as a saving function.

According to (3), (4) and the no-arbitrage condition (7), we obtain $d_t(1 + x) = R_t n k_t + (q_t + \delta_t) a_{t-1}$ and

$$g_t = \frac{x}{1 + x} \left(k_t f'(k_t) + b_t + \xi_t \right) \tag{9}$$

Since g_t only depends on k_t, b_t and ξ_t, we introduce a new function equivalent to (8):

$$k_{t+1} = \mathcal{G}_x (k_t, b_t, \xi_t)$$

The following assumption is satisfied for a very large class of utility and production functions.

Assumption 2. *Function $\mathcal{G} : \mathbb{R}^3_+ \to \mathbb{R}$ is continuously differentiable with*

$$\frac{\partial \mathcal{G}_x}{\partial k_t} > 0, \ \frac{\partial \mathcal{G}_x}{\partial b_t} < 0, \ \frac{\partial \mathcal{G}_x}{\partial \xi_t} > 0$$

Moreover, $\mathcal{G}_x(k, +\infty, \xi) = -\infty$ and $\mathcal{G}_x(k, 0, \xi) > 0$ for any $k > 0$ any $\xi > 0$.

This assumption is for instance satisfied by a Cobb-Douglas production function $F(K, L) = AK^{\alpha}L^{1-\alpha}$ and isoelastic preferences $U(c, d) = \ln c + \beta \ln d$ or $U(c, d) = (c^{1-\sigma} + \beta d^{1-\sigma}) / (1 - \sigma)$ with $\sigma \in (0, 1)$.

Lemma 1. *Under Assumption 2, the sequence $(k_{t+1}, b_t)_{t \geq 0}$ is an interior equilibrium if and only if*

$$k_{t+1} = \mathcal{G}_x(k_t, b_t, \xi_t) \text{ and } b_{t+1} = b_t \frac{f'(k_{t+1})}{n} - \xi_{t+1} \tag{10}$$

with $b_t > 0$ and $k_{t+1} > 0$.

Remark 2. *Budget constraints (2), (3), (4) and (9) imply that*

$$c_t + nk_{t+1} + \frac{b_t}{1 + x} = f(k_t) - \frac{k_t f'(k_t)}{(1 + x)} + \frac{x\xi_t}{(1 + x)} \tag{11}$$

Thus, $b_t \leq (1 + x) f(k_t) + x\xi_t$. Hence, if k_t and ξ_t are bounded from above, the asset value will be also bounded from above.

Let us prove the equilibrium existence before studying its dynamic properties.

Lemma 2 (existence of an interior equilibrium)**.** *Assume that Assumptions 1 and 2 hold. If $f'(0) = \infty$, then the system (10) has a solution (b_t, k_{t+1}) with $b_t > 0$ and $k_{t+1} > 0$ for any $t \geq 0$. Such sequence is an interior equilibrium.*

Remark 3. *This existence result is far from trivial. One may say that, given b_0, it is easy to determine the equilibrium sequence $(k_t, b_t)_{t \geq 1}$ by using (10). However, the difficulty is to ensure that k_t and b_t determined by (10) are positive for every $t > 0$. The existence of an interior equilibrium rests on a sufficiently high productivity of capital ($f'(0) = \infty$). This equilibrium may fail to exist in the case of low productivity. An example of failure with linear technology is provided in Section 6.2 and supplemented with economic interpretation.*[5]

4 Definition and existence of bubbles

In this section, we present a formal definition of bubble and a characterization of bubble existence as its direct consequence.

Solving recursively (7), we obtain an asset price decomposition in two parts

$$q_t = Q_{t,t+\tau} q_{t+\tau} + \sum_{s=1}^{\tau} Q_{t,t+s} \delta_{t+s}, \text{ where } Q_{t,t+s} \equiv \frac{1}{R_{t+1} \dots R_{t+s}}$$

is the discount factor of the economy from date t to $t + s$.

In the spirit of Tirole (1982), Tirole (1985), Kocherlakota (1992), Santos and Woodford (1997) and Huang and Werner (2000), we define the fundamental value of financial asset and the bubble.

[5] See Le Van and Pham (2015) for equilibrium analysis in an infinite-horizon general equilibrium model where the aggregate capital stock k_t may be zero.

Definition 2. *1. The Fundamental Value of a unit of asset at date t is the sum of discounted values of dividends:*

$$FV_t \equiv \sum_{s=1}^{\infty} Q_{t,t+s}\delta_{t+s}$$

2. We say that there is a bubble at date t if $q_t > FV_t$.

3. When $\delta_t = 0$ for any $t \geq 0$ (the Fundamental Value is zero), we say that there is a pure bubble if $q_t > 0$ for any t.

Clearly, we have $q_t = FV_t + \lim_{\tau\to\infty} Q_{t,t+\tau}q_{t+\tau}$. Thus, $q_t - FV_t$ does not depend on t. Therefore, if a bubble exists at date 0, it exists forever.

Remark 4. *1. Our asset is related to the asset with rent (dividend) in Tirole (1985) since both the assets bring dividends at any date. However, Tirole (1985) assumes that the rent (dividend) is stationary while dividends are non-stationary in our model. In Tirole (1985), there is no bubble with a positive rent, while, in our model, asset bubbles may arise as we will show below.*

2. Weil (1990) considers an asset (he calls land) with positive dividends, but in a pure exchange economy, and he assumes that there exists t_0 such that $\delta_t = 0$ for any $t \geq t_0$, while our model encompasses the productive sector and δ_t may be strictly positive at any date.

3. When $\delta_t = 0$ for any t, some others, e.g. Weil (1987), interpret the pure bubble as fiat money.

For notational simplicity, we set $Q_0 \equiv 1$ and $Q_t \equiv Q_{0,t}$ for any t. No-arbitrage condition (7) implies that

$$\begin{aligned} q_0 &= \frac{1}{R_1}(1+\frac{\delta_1}{q_1})q_1 = \frac{1}{R_1R_2}(1+\frac{\delta_1}{q_1})(1+\frac{\delta_2}{q_2})q_2 = \cdots \\ &= Q_Tq_T(1+\frac{\delta_1}{q_1})\cdots(1+\frac{\delta_T}{q_T}) \end{aligned}$$

Bubbles exist if and only if $\lim_{T\to\infty} Q_Tq_T > 0$ which is equivalent to $\prod_{t=1}^{\infty}(1+\frac{\delta_t}{q_t}) < \infty$. Therefore, we have necessary and sufficient conditions (based on endogenous variables) for the existence of bubbles of assets with positive dividends.[6]

Proposition 1. *In the case of strictly positive dividends ($\delta_t > 0$ for any t),[7] the following statements are equivalent.*

1. A bubble exists at date t.

2. $\lim_{T\to\infty} Q_Tq_T > 0$, i.e. $\lim_{T\to\infty} b_Tn^T/\Pi_{\tau=1}^T f'(k_\tau) > 0$.

3. $\sum_{t=0}^{\infty} \delta_t/q_t < \infty$, i.e. $\sum_{t=0}^{\infty} \xi_t/b_t < +\infty$.

[6] Which are similar to those in Montrucchio (2004) or Le Van and Pham (2014).

[7] To ensure a positive asset price ($q_t > 0$) at any date.

Proposition 1 is very general because its proof rests only on the no-arbitrage condition (7) and Definition 2. Here, technology and preferences play no role.

Remark 2 implies $b_t \leq (1+x) f(k_t) + x\xi_t$. Therefore, we can prove a fundamental corollary of point (3) of Proposition 1: bubble existence requires very low dividends with respect to output.

Corollary 1. *Consider the case of positive dividends ($\delta_t > 0$ for any t).*

Bubble existence implies $\sum_{s=1}^{\infty} [\xi_s / f(k_s)] < \infty$. Consequently, at the steady state ($k_t = k$ and $\xi_t = \xi > 0$ for any t), bubbles are ruled out.

Notice that Corollary 1 does not require any condition about the boundedness of capital stock or dividends. It also holds for non-stationary technologies.

Corollary 1 is stronger than a well-known result of literature on rational bubbles in infinite-horizon models (Le Van and Pham, 2014, 2015): bubbles are ruled out if the sequence of ratio of dividend to aggregate output is bounded below from zero.

Let us interpret the asset a_t as land and ξ_t as fruits of land at period t. Thanks to Corollary 1, we realize why Weil (1990) needs to assume that trees produce fruits only for a finite number of periods[8] in order to get land bubbles.

Let us conclude the section by comparing assets with and without dividends

The last point of Corollary 1 means that, at the steady state, an asset yielding positive dividends generates no bubble whatever the level of interest rates. However, bubbles of an asset without intrinsic value (Tirole, 1985) may exist at the steady state when interest rates are low. This is the fundamental difference between assets with and without dividends.

Remark 5. *When there is no bubble, the structure of the asset becomes that of the rent introduced by Tirole (1985). Corollary 1 also reminds us Proposition 7 in Tirole (1985) who considers a model with money in the utility function. Bubble formation rests on transactions and speculative demand for money. Dividends (on money) are reinterpreted by Tirole in terms of (marginal) utility, while, in our paper, asset dividends are paid in consumption units. Tirole (1985) shows that positive returns on money rule out the possibility of bubbles; by contrast, in our model, bubbles may appear when dividends tend to zero (see Sections 6).*

5 Transitional dynamics of capital stocks and asset values

In this section, we provide general results about the equilibrium transition for capital stocks and asset values. According to Lemma 1, the interior equilibrium system is written

$$b_{t+1} = b_t \frac{f'(k_{t+1})}{n} - \xi_{t+1} \text{ and } k_{t+1} = \mathcal{G}_x(k_t, b_t, \xi_t) \tag{12}$$

with

$$b_t > 0 \text{ and } k_{t+1} > 0 \tag{13}$$

(12) is a two-dimensional system with the degree of forward altruism x and an infinite number of other parameters (the sequence of exogenous dividends (ξ_t)). Systems of this kind are difficult to handle. Nevertheless, there is something to say about equilibrium existence (see Lemma 2).

[8]Formally, there is T_0 such that $\xi_t = 0$ for any $t \geq T_0$.

First, we look to the set of equilibrium trajectories and, then, we give some asymptotic results. We observe that, for each $b_0 > 0$, there exists a unique sequence $(k_t, b_t)_{t\geq 0}$ satisfying (12). So, given an equilibrium (k_{t+1}, b_t), the asset fundamental value FV_0 at date 0 can be computed through b_0. Hence, we write $FV_0 = FV_0(b_0)$. The initial asset value b_0 affects the size of bubbles $b_0 - FV_0(b_0)$ along the equilibrium transition, and indeterminacy of initial bubble entails in turn the multiplicity of bubbly equilibria. The following lemma is one of the main contributions of the paper.

Lemma 3. *1. The set $\mathcal{B}_0$ of all the values $b_0 > 0$ such that the sequence $(k_{t+1}, b_t)_{t\geq 0}$ determined by (12) is an equilibrium, is an interval.*

2. The fundamental value function $FV_0(b_0)$ is decreasing in b_0 while the size of bubble $b_0 - FV_0(b_0)$ is strictly increasing.

3. There exists at most one bubbleless solution. Moreover, if there are two equilibria with initial asset values $b_{0,1} < b_{0,2}$, then any equilibrium with initial asset value $b_0 \in (b_{0,1}, b_{0,2}]$ is bubbly.

According to (12), it is easy to see that $k_{t+1} \leq \mathcal{G}_x(k_t, 0, \bar{\xi})$, where $\bar{\xi} \equiv \sup_t \xi_t$.

Assumption 3. *There exists a unique $k_{\bar{\xi},x} > 0$ such that $k_{\bar{\xi},x} = \mathcal{G}_x(k_{\bar{\xi},x}, 0, \bar{\xi})$ with $\mathcal{G}_x(k, 0, \bar{\xi}) > k$ if $k < k_{\bar{\xi},x}$ and $\mathcal{G}_x(k, 0, \bar{\xi}) < k$ if $k > k_{\bar{\xi},x}$.*

Under this assumption, it is easy to see that $k_t < \max(k_0, k_{\bar{\xi},x})$ for any t. So, (k_t) is uniformly bounded from above. Therefore, Corollary 1 leads to the following result.

Corollary 2. *Under Assumption 3, bubbles exist only if $\sum_{s=1}^{\infty} \xi_s < \infty$. Consequently, at the steady state ($k_t = k$ and $\xi_t = \xi > 0$ for any t), bubbles are ruled out.*

Hence, when $\sum_{s=1}^{\infty} \xi_s = \infty$ there is no bubbly equilibrium, and then, according to point (3) of Lemma 3, there exists a unique equilibrium. Thanks to Corollary 3, we need to focus only on the case $\sum_{s=1}^{\infty} \xi_s < \infty$ to look for economies where bubbles may arise.

Assumption 4. *For b and ξ small enough, there exists a unique $k_{b,\xi}$ solution to $\mathcal{G}_x(k, b, \xi) = k$.*

Denote by k_x^* the solution to $\mathcal{G}_x(k, 0, 0) = k$. Observe that k_x^* is the level of capital stock at the steady state of the economy without financial asset ($b_t = 0$ and $\xi_t = 0$ for any t). Notice also that $\lim_{b,\xi\to 0} k_{b,\xi} = k_x^*$ and $k_{b,\xi}$ is decreasing in b.

Lemma 4. *Let Assumptions 1, 2, 3 and 4 hold. Suppose also that $f'(k_x^*) > n$ and $\lim_{t\to\infty} \xi_t = 0$. Then, $\lim_{t\to\infty} b_t = 0$.*

The intuition behind this lemma is straightfoward. When interest rates are sufficiently high (that is $f'(k_x^*) > n$) and asset dividends tend to zero, households prefer to invest more in physical capital and less in financial asset. Accordingly, the value of the financial asset converges to zero.

Let us present the main result of the section: the global analysis of dynamics of capital stocks and asset values.

Proposition 2. *Let Assumptions 1, 2, 3, 4 hold.*

1. If $f'(k_x^*) > n$, then there exists a unique equilibrium, which is bubbleless.

In addition, if $\lim\limits_{t\to\infty} \xi_t = 0$, then $\lim_{t\to\infty} b_t = 0$.

2. *If $f'(k_x^*) < n$ and $\xi_0 \geq \xi_1 \geq \cdots \geq \lim_{t\to\infty} \xi_t = 0$. Denote by x_n the solution to $f'(x) = n$. Then, any equilibrium belongs to one of the following three cases.*

   (a) $\liminf_{t\to\infty} k_t < x_n$. *In this case, the equilibrium solution is bubbleless and unique.*

   (b) $\lim_{t\to\infty} k_t = k_x^*$ *and* $\lim_{t\to\infty} b_t = 0$.

   (c) $\lim_{t\to\infty} k_t = x_n$ *and* $\lim_{t\to\infty} b_t = b_n$ *where* b_n *satisfies* $x_n = \mathcal{G}_x(x_n, b_n, 0)$.

Proposition 2 can be viewed as a generalized version of Proposition 1 in Tirole (1985). The novel point is that we work with non-stationary dividends that rise a challenge, while Tirole (1985) considers an asset with zero dividend (he calls it *bubbles*).[9] Another added value is the role of altruism which we will discuss in more detail in Section 6.1.

Let us provide intuition of the first part of Proposition 2. Recall that the size of bubble is the discounted value of one unit of asset at the infinity

$$\lim_{T\to\infty} Q_T q_T = \lim_{T\to\infty} \frac{1}{a_0} Q_T b_T n^T = \lim_{T\to\infty} \frac{1}{a_0} \frac{n^T}{\Pi_{\tau=1}^T f'(k_\tau)} b_T$$

This size depends on the asset price and the discount factors of the economy. Since the asset value is uniformly bounded from above, and interest rates are high (in the sense that $f'(k_x^*) > n$), the value of bubble will be zero. This is true whatever the level of dividends. Considering a particular case where $\xi_t = 0$ for any t and no altruism ($x = 0$), we recover point (a) of Proposition 1 in Tirole (1985). However, in general, along the unique equilibirium, the asymptotic property of capital stocks and asset values may not hold. There is room for fluctuations in the capital stocks if dividends ξ_t fluctuate.[10]

The second case ($f'(k_x^*) < n$) is much more complicated because of the multiple equilibria arising. However, we get also a novel result: if an equilibrium experiences a bubble, then capital stock and asset value must converge. Asset values may converge to zero or to a positive value.

The following result concludes the section and is a direct consequence of Proposition 2 and Lemma 1.

Corollary 3. *Let Assumptions 1, 2, 3 and 4 hold. The economy experiences a bubble only if $f'(k_x^*) \leq n$ and $\sum_{t\geq 1} \xi_t < \infty$.*

The next section illustrates by means of examples the theoretical results in the second part of Proposition 2, the case of multiple equilibria (either with or without bubbles).

6 Examples

In this section, we consider some particular cases and provide more explicit equilibrium analysis. We give also some examples of multiple equilibria with and without bubbles.

6.1 Logarithmic utility and Cobb-Douglas technology

We consider the case of a Cobb-Douglas production function $f(k) = Ak^\alpha$ and a logarithmic utility function $U(c, d) = \ln c + \beta \ln d$ with $\beta > 0$. The income sharing between consumption

[9] Tirole (1985) also considers another asset that bring stationary dividend (or rent). However, he assumed that there is no bubble on this asset.

[10] Refer to Le Van and Pham (2015) for an analysis in infinite-horizon setting.

and total saving is given by

$$c_t = \frac{1}{1+\beta}\,(w_t + g_t) \quad \text{and} \quad s_t + q_t a_t = \frac{\beta}{1+\beta}\,(w_t + g_t)$$

The equilibrium system is explicitly written

$$nk_1 + b_0 = \frac{\beta}{1+\beta}\,(w_0 + g_0)$$

$$k_{t+1} = \frac{\alpha A \gamma_x k_t^{\alpha} + (1-\sigma)\,\xi_t - \sigma b_t}{n} \quad \forall t \geq 1 \tag{14}$$

$$b_{t+1} = \frac{\alpha A b_t}{n k_{t+1}^{1-\alpha}} - \xi_{t+1} \tag{15}$$

with the following parameters indexed in the degree of altruism (x):

$$\gamma_x \equiv \frac{\beta}{1+\beta}\frac{1-\alpha+x}{\alpha\,(1+x)},\ \theta_x^* \equiv \frac{\alpha\,(\gamma_x - 1)}{\sigma} \text{ and } \sigma \equiv 1 - \frac{\beta}{1+\beta}\frac{x}{1+x}$$

With our explicit production and utility function, we compute the reduced functions:

$$\mathcal{G}_x(k, b_t, \xi_t) = \frac{\alpha A \gamma_x k^{\alpha} + (1-\sigma)\,\xi_t - \sigma b_t}{n}$$

$$\mathcal{G}_x(k, 0, 0) = \frac{\alpha A \gamma_x}{n} k^{\alpha}, \quad k_x^* \equiv (\alpha A \gamma_x / n)^{1/(1-\alpha)}$$

Remark 6. *1. We observe that $\gamma_x = n/f'(k_x^*)$, so condition $f'(k_x^*) < n$ becomes equivalent to $\gamma_x > 1$. Parameter γ_x captures the distorsion with respect to the Golden Rule.*

2. Under Cobb-Douglas technology and Assumption 1, we see that (k_t) is uniformly bounded from above.

It is easy to check that these specifications satisfy Assumptions 2, 3, 4. Consequently, Proposition 2 applies. Moreover, according to Corollary 2, an equilibrium is bubbly only if $\sum_{t\geq 1} \xi_t < \infty$, the case we will focus on. The following result complements Proposition 2.

Proposition 3. *Assume that $f(k) = Ak^{\alpha}$ and $U(c,d) = \ln c + \beta \ln d$ with $0 < \beta < 1$. Suppose also that $\xi_t > 0$ for any t and $\lim_{t\to\infty} \xi_t = 0$.*

1. *If $\gamma_x \leq 1$ (i.e., $f'(k_x^*) \geq n$). If (b_t, k_{t+1}) is an equilibrium with $\inf_{t\geq 0} k_t > 0$, then $\lim_{t\to\infty} b_t/(Ak_t^{\alpha}) = 0$.*

2. *If $\gamma_x > 1$ (i.e., $f'(k_x^*) < n$). If (b_t, k_{t+1}) is an equilibrium with $\inf_{t\geq 0} k_t > 0$, then there are two cases.*

   (a) The sequence (b_t) converges to $b = 0$, and (k_t) converges to $k_x^* \equiv (\alpha A \gamma_x / n)^{1/(1-\alpha)}$.

   (b) The sequence (b_t) converges to $b = n\,(\gamma_x - 1)\,x_n$, and (k_t) converges to $x_n \equiv (\alpha A/n)^{1/(1-\alpha)}$.

Remark 7 (comparative statics). *(i) The limit of capital stock. The limit k_x^* in case 2.a. of Proposition 3 increases in the degree of forward altruism (x).*

(ii) The limit of asset value. The limit $b = n(\gamma_x - 1)x_n$ in case 2.b. of Proposition 3 increases in x.

These positive effects are very intuitive and depend on the definition of forward altruism: bequests are proportional to consumption of old and they improve income, and then saving of young people. The more the savings of the young, the higher the amount at their disposal to buy the financial asset and/or the physical capital.

In Proposition 3, we consider only equilibria with $\inf_t k_t > 0$, which is true in most of the cases. However, there are cases where k_t converges to zero: in the following example, there is an equilibrium with $\lim_{t\to\infty} k_t = 0$.

Example 1 (equilibrium with $\lim_{t\to\infty} k_t = 0$)**.** *Consider the selfish economy ($x = 0$). Assume that $f(k) = Ak^\alpha$ and $U(c,d) = \ln c + \beta \ln d$ with $0 < \beta < 1$.*

Let

$$\lambda \equiv \frac{\alpha^2 + \sqrt{4\alpha^3 + \alpha^4}}{1-\alpha} > \max\left\{1, \ln\left(1 + \frac{2}{\gamma_0}\right)\right\}$$

and an explicit sequence $x_t \equiv \max\left\{e^{\lambda^t}, 1 + 2/\gamma_0\right\}$, where

$$\gamma_0 \equiv \frac{1-\alpha}{\alpha}\frac{\beta}{1+\beta}$$

Define a sequence (b_t, k_{t+1}, ξ_t) with

$$b_t = \alpha A \gamma_0 k_t^\alpha - n k_{t+1} \text{ and } k_{t+1} = \frac{\alpha A k_t^\alpha}{n x_t}$$

$$\xi_{t+1} \equiv \frac{\alpha A b_t}{n k_{t+1}^{1-\alpha}} - b_{t+1} \tag{16}$$

Then, (b_t, k_{t+1}) is an interior equilibrium with $\lim_{t\to\infty} k_t = \lim_{t\to\infty} b_t = 0$.

Focus now on the structure of dividends to find a sufficient condition for multiple bubbly equilibria. Dividends decrease geometrically but their aggregate value is required to be constant over time and bounded.

Example 2 (continuum of bubbly equilibria with forward altruism)**.** *Let $\xi_t \equiv \xi/n^t$ with $n > \gamma_x > 1$ and*

$$k_m \equiv \min\{k_0, x_n\} \leq x_n \equiv \left(\frac{\alpha A}{n}\right)^{\frac{1}{1-\alpha}} < \bar{k} \equiv \left(\frac{\alpha A \gamma_x}{n}\right)^{\frac{1}{1-\alpha}} \leq k_M \equiv \max\left\{k_0, \bar{k}\right\}$$

$$\xi \in (0, \bar{\xi})$$

where $\bar{\xi} > 0$ is solution to

$$\frac{\alpha}{\alpha\gamma_x + (1-\sigma)\xi/(Ak_M^\alpha)} = \frac{1}{n} + \frac{\xi}{\theta_x^* A k_m^\alpha}$$

Then, any sequence (b_t, k_{t+1}) determined by the system (14)-(15) and b_0 such that $\theta_x^* A k_0^\alpha / n < b_0 < \theta_x^* A k_0^\alpha$, is an equilibrium.

Remark 8. *Bubbles arise in an OLG model à la Diamond (Tirole, 1985). However, an arbitrarily small degree of altruism à la Barro (1974) immediately kills the bubble in models à la Diamond (Bosi et al., 2015).*

In our paper, forward altruism is based on constraints instead of utility. In this case, bubbles are preserved in overlapping generations models with altruism. The reason is that bequests from old to young are proportional to consumption of old. The old finance these bequests partly purchasing the bubble when young.

6.1.1 Explicit solution in the case of pure bubble

In this section, we consider the dynamics of pure bubbles à la Tirole (1985) by setting $\xi_t = 0$ for any t. In this case, the value of bubble equals the asset value. We provide the explicit trajectories of both capital stocks and asset values.

The equilibrium system is written

$$nk_1 + b_0 = \frac{\beta}{1+\beta}(w_0 + g_0) \tag{17}$$

$$nk_{t+1} + \sigma b_t = \gamma_x \alpha A k_t^{\alpha} \forall t \geq 1 \tag{18}$$

$$nb_{t+1} = \alpha A k_{t+1}^{\alpha-1} b_t \tag{19}$$

with $k_{t+1} > 0$, $b_t \geq 0$, where

$$\sigma \equiv 1 - \frac{\beta}{1+\beta}\frac{x}{1+x} \in (0,1], \quad \gamma_x \equiv \frac{\beta}{1+\beta}\frac{1-\alpha+x}{\alpha(1+x)} = \frac{n}{f'(k_x^*)}$$

Here, k_x^* is the capital intensity in the bubbleless steady state, that is the steady state solution of (18) with $b = 0$:

$$k_x^* = \rho_{\gamma_x}^{1/(1-\alpha)} \tag{20}$$

with $\rho_{\gamma_x} \equiv \gamma_x \alpha A/n$. We eventually introduce the bubble critical value:

$$\bar{b}_x \equiv (w_0 + g_0)\frac{\beta}{1+\beta}\frac{\gamma_x - 1}{\gamma_x - 1 + \sigma} = (w_0 + g_0)\left[1 - \frac{1+x+\alpha\beta}{(1+x)(1-\alpha)(1+\beta)}\right]$$

which is positive if $\gamma_x > 1$.

These elements allows us to introduce the main result of this section.

Proposition 4. *Assume that $f(k) = Ak^{\alpha}$, $U(c,d) = \ln c + \beta \ln d$ with $0 < \beta < 1$, and $\xi_t = 0$ for any t.*

1. *If $\gamma_x \leq 1$ (i.e. $f'(k_x^*) \geq n$), the equilibrium is unique and bubbleless and the equilibrium sequence of capital intensities is given by*

$$k_t = \rho_{\gamma_x}^{\frac{1-\alpha^{t-1}}{1-\alpha}} k_1^{\alpha^{t-1}} \quad \forall t \geq 2, \quad k_1 = \frac{\beta}{n(1+\beta)}(w_0 + g_0) \tag{21}$$

   Moreover, $\lim_{t\to\infty} k_t = k_x^*$, where k_x^* is given by (20).

2. *If $\gamma_x > 1$ (i.e. $f'(k_x^*) < n$), the equilibrium is indeterminate. The set of equilibria $(k_{t+1}, b_t)_{t\geq 0}$ is defined by (18), (19), and $b_0 \in [0, \bar{b}_x]$. Moreover,*

   (a) *(bubbleless equilibrium) If $b_0 = 0$, and, thus, $b_t = 0$ forever. The sequence (k_t) is given by (21).*

   (b) *(bubbly equilibrium) If $b_0 > 0$, then $b_t > 0$ for any t.*
   When $b_0 < \bar{b}_x$, we have $\lim_{t\to\infty} b_t = 0$ and $\lim_{t\to\infty} k_t = k_x^*$.
   When $b_0 = \bar{b}_x$, we have $\lim_{t\to\infty} b_t > 0$. We also have

$$b_t = \frac{\gamma_x - 1}{\sigma} nk_{t+1} \forall t \geq 0 \tag{22}$$

$$k_t = \rho_1^{\frac{1-\alpha^{t-1}}{1-\alpha}} k_1^{\alpha^{t-1}} \quad \forall t \geq 2, \quad k_1 = \frac{\alpha(w_0 + g_0)}{n(1-\alpha)}\left(1 - \frac{\beta}{1+\beta}\frac{x}{1+x}\right) \tag{23}$$

   and $\rho_1 \equiv \alpha A/n$. Moreover,

$$\lim_{t\to\infty} k_t = \rho_1^{1/(1-\alpha)} < k_x^* \text{ and } b_x \equiv \lim_{t\to\infty} b_t = n\frac{\gamma_x - 1}{\sigma}\rho_1^{1/(1-\alpha)} > 0. \tag{24}$$

Definition 3. $\bar{b}_x$ *is the (upper) size of bubbly asset value at initial date with forward altruism (in the case* $\gamma_x > 1$*).*

The value $\rho_1^{1/(1-\alpha)}$ corresponds to the value x_n determined by $f'(x_n) = n$ and introduced in Proposition 2.

Proposition 4 illustrates and complements Proposition 2 in the case $\xi_t = 0$. It is instructive to compare these two propositions. Proposition 4 supplies a number of new results: explicit equilibrium sequences, a proof of global convergence, a necessary and sufficient condition for bubble existence as well as for equilibrium indeterminacy. All these issues remain unaddressed in theoretical papers.[11]

Beyond the methodological interest of an alternative definition of altruism, the novelty of this section rests on the expression of $\bar{b}_x$, the maximum feasible bubble at the initial date, in terms of fundamental parameters. The explicit form also allows us to compute the impact of some relevant parameter (impatience and altruism) on equilibrium trajectories.

Comparative statics

1. (existence of bubble). Condition $\gamma_x \equiv n/f'(k_x^*) > 1$ (i.e. low interest rates or capital overaccumulation) is equivalent to

$$\frac{\alpha(1+x)}{1-\alpha+x} < \frac{\beta}{1+\beta} \tag{25}$$

   The left-hand side of (25) decreases with x. Thus, forward altruism promotes the emergence of bubbles.

2. *Both limits* k_x^* *and* b_x increase in x. The intuition is similar to that in Remark 7.

3. (maximum value $\bar{b}_x$). Let us compute the effects of initial endowments, patience and altruism on the maximum level of asset value. Equation (**??**) implies that

$$\frac{\partial \bar{b}_x}{\partial k_0}, \frac{\partial \bar{b}_x}{\partial \beta}, \frac{\partial \bar{b}_x}{\partial x} > 0$$

4. (equilibrium transition). Consider the case of low interest rates (ie. $f'(k_x^*) < n$ or $\gamma_x > 1$). Look at the asymptotically bubbly equilibrium (i.e. $b_0 = \bar{b}_x$). We see that $b_0 = \bar{b}_x$ increases in x, so k_1 determined by (17) decreases in x. Since $k_{t+1} = \rho_1 k_t$ for any t, we see that k_t decreases in x for any t. Hence, R_t increases in x for any t. By using the induction argument and the fact that $b_t = R_t b_{t-1}/n$ for any $t \geq 0$, we obtain that b_t increases in x for any t. So, along the asymptotically bubbly equilibrium, asset value[12] b_t increases but capital stock k_t decreases in the forward altruism degree.

6.2 Logarithmic utility and linear technology

We consider the case of a linear production function $F(K, L) = RK + wL$ and logarithmic utility function $U(c_t, d_{t+1}) = \ln c_t + \beta \ln d_{t+1}$ with $0 < \beta < 1$. In this case, the incoe sharing

[11] Bosi and Seegmuller (2013) show the local indeterminacy of real bubbles (rational exuberance). We focus instead on global indeterminacy of real bubbles.

[12] When dividends are zero, asset value and bubble coincide.

is written

$$c_t = \frac{1}{1+\beta}(w_t + g_t) \text{ and } s_t + q_t a_t = \frac{\beta}{1+\beta}(w_t + g_t)$$

and

$$g_t = \frac{x}{1+x}(k_t f'(k_t) + b_t + \xi_t) \ \forall t \geq 1$$

(g_0 is given). The equilibrium system becomes

$$nk_{t+1} + (1 - \frac{\beta}{1+\beta}\frac{x}{1+x})b_t = R\frac{\beta}{1+\beta}\frac{x}{1+x}k_t + \frac{\beta}{1+\beta}\left(w + \frac{x}{1+x}\xi_t\right) \ \forall t \geq 1 \tag{26}$$

$$b_{t+1} + \xi_{t+1} = \frac{R}{n}b_t \tag{27}$$

with $k_t, b_t > 0$. Notice that, at the initial date,

$$nk_1 + b_0 = \frac{\beta}{1+\beta}(w + g_0)$$

We compute the fundamental value of financial asset:

$$FV_0 = \sum_{t=1}^{\infty}\frac{\delta_t}{R^t} = \sum_{t=1}^{\infty}\frac{n^t\xi_t}{R^t a_0}$$

Solving recursively no-arbitrage condition in (27) yields

$$b_t = \frac{R^t}{n^t}\left(b_0 - \sum_{s=1}^{t}\frac{n^s}{R^s}\xi_s\right) \tag{28}$$

Proposition 5. *Assume that $F(K, L) = RK + wL$ and $U(c, d) = \ln c + \beta \ln d$ with $0 < \beta < 1$. At equilibrium, we have*

$$nk_{t+1} + b_t = D^t\frac{\beta}{1+\beta}(w + g_0) + \frac{\beta w}{1+\beta}\frac{1-D^t}{1-D} \text{ where } D \equiv \frac{R}{n}\frac{\beta}{1+\beta}\frac{x}{1+x} \tag{29}$$

Hence,

1. *$R > n$. There is no bubbly equilibrium.*
2. *$R \leq n$. Assume that*

$$D^t\frac{\beta}{1+\beta}(w + g_0) + \frac{\beta w}{1+\beta}\frac{1-D^t}{1-D} - \frac{R^t}{n^t}\left[\frac{\beta}{1+\beta}(w + g_0) - \sum_{s=1}^{t}\frac{n^s}{R^s}\xi_s\right] > 0 \ \forall t \geq 1$$

Then, any interior equilibrium $(k_{t+1}, b_t)_{t\geq 0}$ is determined by (28), (29) and

$$b_0 \in \left[a_0 FV_0, \frac{\beta}{1+\beta}(w + g_0)\right)$$

If $b_0 > a_0 FV_0$, then the equilibrium is bubbly: in this case, $\lim_{t\to\infty} b_t > 0$ if and only if $R = n$.

Condition $R > n$ corresponds to the case of high interest rate in Proposition 2.

Remark 9 (no interior equilibrium). *According to (29), we see that*

$$D^t \frac{\beta}{1+\beta}(w+g_0) + \frac{\beta w}{1+\beta}\frac{1-D^t}{1-D} > b_t = \frac{R^t}{n^t}\left(b_0 - \sum_{s=1}^{t}\frac{n^s}{R^s}\xi_s\right) \geq \sum_{s=t+1}^{\infty}\frac{n^{s-t}}{R^{s-t}}\xi_s$$

Hence, there is no interior equilibrium if

$$D^t \frac{\beta}{1+\beta}(w+g_0) + \frac{\beta w}{1+\beta}\frac{1-D^t}{1-D} < \sum_{s=t+1}^{\infty}\frac{n^{s-t}}{R^{s-t}}\xi_s \ \forall t$$

This happens when the productivity R is low. The intuition is that, when the productivity is low, households tend to invest in financial asset rather than in physical capital. Therefore, the capital stock k_t may be zero.

7 Bubble, asset price and interest rate

7.1 Does the existence of bubbles really require low interest rates and low dividends?

The seminal article by Tirole (1985) proves that pure bubbles may arise if the interest rate at the steady state of the economy without financial asset is below the population growth rate. As shown above, this result still holds for an asset brining non-stationary dividends in an altruistic economy. Both findings are based on the boundedness of total outputs (per capita): $f(k_t) + \xi_t$.

In this section, we revisit this result. Precisely, we consider an economy where the output may grow, and we wonder whether existence of bubble still requires low interest rates and dividends conditions.

For the sake of simplicity, we reconsider Proposition 5 but with a non-stationary linear technology: $F_t(K, L) = RK + w_t L$, where $R, w_t > 0$ are exogenous. The equilibrium system becomes

$$nk_{t+1} + (1 - \frac{\beta}{1+\beta}\frac{x}{1+x})b_t = R\frac{\beta}{1+\beta}\frac{x}{1+x}k_t + \frac{\beta}{1+\beta}\left(w_t + \frac{x}{1+x}\xi_t\right) \ \forall t \geq 1 \tag{30}$$

$$b_{t+1} + \xi_{t+1} = \frac{R}{n}b_t$$

We have, as (28),

$$b_t = \frac{R^t}{n^t}\left(b_0 - \sum_{s=1}^{t}\frac{n^s}{R^s}\xi_s\right) \tag{31}$$

Let us provide an example where (1) bubbles appear, (2) $R > n$ and (3) ξ_t is unbounded. To do so, we choose (w_t, ξ_t) and b_0 such that

$$\frac{\beta}{1+\beta}w_t > \left(1 - \frac{\beta}{1+\beta}\frac{x}{1+x}\right)\frac{R^t}{n^t}\left(b_0 - \sum_{s=1}^{t}\frac{n^s}{R^s}\xi_s\right) \tag{32}$$

$$b_0 \in \left[\sum_{s=1}^{\infty}\frac{n^s}{R^s}\xi_s, \frac{\beta}{1+\beta}(w+g_0)\right] \tag{33}$$

Then the sequence (k_{t+1}, b_t) defined by (30), (31), (32) and (33) is an interior equilibrium with bubble. The intuition is that, even when the interest rate is high (i.e. $R > n$), the growth of asset values does not violate the equilibrium system and the positivity constraints because the productivity of the capital-free side fo production w_t grows faster than the rate $(R/n)^t$.

Remark 10. *Weil (1990) considers on OLG model where consumers receive exogenous dividends. However, Weil (1990) (page 1469) assumes that dividends become zero from some date on to allow for the possibility of bubble. However, the above example shows that a bubbly equilibrium is possible even if dividends are positive at any date and may tend to infinity.*

7.2 Bubbles and monotonicity of asset prices

By definition, an asset bubble appears when the asset price is strictly higher than the asset fundamental value. Some authors are interested in checking whether a causal link holds between the existence of asset bubble and the rise of asset price. Weil (1990) explains why along a bubbly equilibrium the asset prices may decrease. In Proposition 5, under a linear technology, the asset price at date 0 is given by

$$q_t = \frac{b_t}{a_t} = \frac{b_t n^t}{a_0} = \frac{R^t}{a_0}\left(b_0 - a_0 FV_0 + \sum_{s=t+1}^{\infty} \frac{n^s \xi_s}{R^s}\right)$$

We see that the asset price q_t may increase or decrease or even fluctuate (in time) along a bubbly equilibrium. In other words, there is no causal relationship between the existence of bubbles and monotonicity of asset pricing.

8 Conclusion

We have introduced two new ingredients in an overlapping generations model à la Diamond (1965): an asset bringing positive dividends and a kind of descendent altruism. We have shown that bubbles are ruled out if the sum (over time) of ratios of dividend to output is finite. When outputs are bounded from above, the economy experiences a bubble only when (1) interest rates remain below the population growth factor and (2) the sum (over time) of dividends is finite. Some examples of multiple bubbly equilibria have been provided. However, when outputs are not bounded, bubbles may appear even if the interest rates are greater than the population growth rates or even if dividends do not converge to zero (or even if they tend to infinity).

In standard framework, the forward altruism promotes pure bubble à la Tirole (1985) and has a positive impact on asset values but a negative impact on the capital stocks along the transition sequence of an asymptotically bubbly equilibrium.

9 Appendix: proofs of Section 3

Proof of Lemma 2. First, we prove that the following claim.

Claim: For each t, given numbers $b_{t+1}, k_t > 0$ and positive sequence (ξ_{t+1}), there exists b_t and k_{t+1} such that

$$k_{t+1} = \mathcal{G}_x(k_t, b_t, \xi_t), \quad b_{t+1} = \frac{b_t f'(k_{t+1})}{n} - \xi_{t+1}$$
$$b_t > 0, \quad k_{t+1} > 0.$$

It is sufficient to prove that there is $k_{t+1} > 0$ such that

$$k_{t+1} - \mathcal{G}_x\left(k_t, \frac{n\left(b_{t+1}+\xi_{t+1}\right)}{f'\left(k_{t+1}\right)}, \xi_t\right) = 0 \tag{34}$$

According to Assumption 1, the left-hand side of the above equation is a decreasing function on k_{t+1}. Moreover, it is negative when k_{t+1} is small enough and positive when k_{t+1} is high enough. Therefore, equation (34) has a unique solution $k_{t+1} > 0$. For such $k_{t+1} > 0$, we determine b_t by

$$b_t = \frac{n}{f'\left(k_{t+1}\right)}\left(b_{t+1}+\xi_{t+1}\right)$$

It is easy to see that b_t is uniformly bounded from above when k_t tends to zero. Indeed, we have

$$k_{t+1} = \mathcal{G}_x\left(k_t, \frac{n\left(b_{t+1}+\xi_{t+1}\right)}{f'\left(k_{t+1}\right)}, \xi_t\right) \geq 0$$

Therefore, by combining with $\mathcal{G}_x(k_t, +\infty, \xi_t) = -\infty$, we obtain that b_t is uniformly bounded from above when k_t tends to zero.

We now come back to the proof of Lemma 2. Consider the following T-truncated system with $T > 0$, that is $b_{T+1} = 0$ and, for any $t \leq T$,

$$k_{t+1} = \mathcal{G}_x\left(k_t, b_t, \xi_t\right), \quad b_{t+1} = \frac{b_t f'(k_{t+1})}{n} - \xi_{t+1}$$

with $b_t > 0$ and $k_{t+1} > 0$. According to the previous result, there exist $b_T > 0$ and $k_{T+1} > 0$ such that

$$k_{T+1} = \mathcal{G}_x\left(k_T, b_T, \xi_T\right), \quad b_T = \frac{n}{f'\left(k_{T+1}\right)}\xi_{T+1}.$$

By using the induction argument, we conclude that the T-truncated system has a solution $(b_t^T, k_{t+1}^T)_{t \leq T}$.

Let now T tend to infinity: there exists a sub-sequence (t_n) such that $\lim_{n\to\infty}\left(b_t^{t_n}, k_{t+1}^{t_n}\right) = (b_t, k_{t+1})$ for any t. It is easy to see that $(b_t, k_{t+1})_{t\geq 0}$ is a solution to (12). □

Proof of Corollary 1. Condition (11) implies that $b_t \leq (1+x) f(k_t) + x\xi_t$. By combining this with point (iii) of Proposition 1, the existence of bubble implies that

$$\sum_{t=1}^{\infty} \frac{\xi_t}{(1+x) f(k_t) + x\xi_t} < \infty$$

Since $x > 0$, we have $\lim_{t\to\infty} \xi_t / f(k_t) = 0$. So, for any t high enough, we have

$$\frac{\xi_t}{(1+x) f(k_t) + x\xi_t} > \frac{\xi_t}{(2+x) f(k_t)}$$

As a result, we obtain

$$\sum_{s=t}^{\infty} \frac{\xi_s}{f\left(k_s\right)} < (2+x) \sum_{s=t}^{\infty} \frac{\xi_t}{(1+x) f(k_t) + x\xi_t} < \infty.$$

□

Proof of Lemma 3. (1) Consider the two solutions $b_0^1 \leq b_0^2$ with (b_t^1, k_{t+1}^1) and (b_t^2, k_{t+1}^2) two corresponding sequences of asset values and capital stocks. Suppose that $b_0^1 \leq b_0 \leq b_0^2$. Consider the sequence (b_t, k_{t+1}) generated (12). By induction, it is easy to prove that for any $t \geq 0$, we have $k_{t+1}^1 \geq k_{t+1} \geq k_{t+1}^2$ and $b_t^1 \leq b_t \leq b_t^2$. Hence, the sequence (b_t, k_{t+1}) is also a solution of the dynamic system.

(2) Take b_0^1 and b_0^2 as at point (1). For any t, we have $k_{t+1}^1 \geq k_{t+1}^2$ and $b_t^1 \leq b_t^2$, and, therefore, $f'(k_{t+1}^1) \leq f'(k_{t+1}^2)$. Hence, $FV(b_0^1) \geq FV(b_0^2)$ and, if $b_0^1 < b_0^2$, we have $b_0^1 - FV(b_0^1) < b_0^2 - FV(b_0^2)$. The function $b_0 - FV(b_0)$ is strictly increasing.

(3) Since, for any solution, we have $b_0 - FV(b_0) \geq 0$, point (3) is a direct consequence of point (2). □

10 Appendix: proofs of Section 5

Proof of Lemma 4. Fix $\bar{x} > k_x^*$ such that $f'(\bar{x}) > n$.

Let $\xi > 0$. There exists $T(\xi)$ such that $\xi_t \leq \xi$ for any $t \geq T(\xi)$. Hence, $k_{T+t} \leq \mathcal{G}_x^t(k_T, 0, \xi)$. Moreover, we see that $\lim_{t\to\infty} \mathcal{G}_x^t(k_T, 0, \xi) = k_{0,\xi}$ for any $k_T > 0$. Hence, we obtain $\limsup_{t\to\infty} k_t \leq k_{0,\xi}$ for any ξ.

Let ξ converge to 0, we have that $k_{0,\xi}$ converges to k_x^*, and hence $\limsup_{t\to\infty} k_t \leq k_x^* < \bar{x}$. So, there exists T high enough such that $k_t \leq \bar{x}$ for any $t \geq T$.

Assume that $\limsup_t b_t > 0$. Let $\epsilon > 0$ satisfy $(\limsup_t b_t)\,[f'(\bar{x})/n - 1] - \epsilon > 0$. Then, there exists $T_0 > T$ high enough such that $k_t \leq \bar{x}$ and $\xi_t < \epsilon$ for any $t \geq T$. Thus, we have

$$b_{T+1} - b_T = b_T \left[\frac{f'(k_{T+1})}{n} - 1\right] - \xi_{T+1} \geq b_T \left[\frac{f'(\bar{x})}{n} - 1\right] - \epsilon > 0$$

Therefore, $b_{T+1} > b_T$, and hence

$$\begin{aligned} b_{T+2} - b_{T+1} &= b_{T+1} \left[\frac{f'(k_{T+2})}{n} - 1\right] - \xi_{T+2} \\ &\geq b_{T+1} \left[\frac{f'(\bar{x})}{n} - 1\right] - \epsilon > b_T \left[\frac{f'(\bar{x})}{n} - 1\right] - \epsilon > 0 \end{aligned}$$

So, the sequence $(b_t)_{t\geq T}$ is increasing and converges to $\bar{b} < +\infty$, since from (11), $(b_t)_t$ is uniformly bounded from above. Therefore, from the no-arbitrage condition (7), k_{t+1} converges to x_n with $f'(x_n) = n$. This leads to a contradiction since, for all $t \geq T$, $f'(k_t) \geq f'(\hat{x}) > n$. We have proved that b_t converges to 0. □

Proof of Proposition 2. Part 1. Assume that $f'(k_x^*) > n$.

Case 1: $\limsup_{t\to\infty} \xi_t > 0$, according to Corollary 1, every equilibrium is bubbleless. Point (iii) of Lemma 3 implies that the uniqueness of equilibrium.

Case 2: $\lim_{t\to\infty} \xi_t = 0$. For any $\xi > 0$, there exists T such that $\xi_{T+t} < \xi$ for any $t \geq 0$. Hence, $k_{T+t} \leq \mathcal{G}_x^t(k_T, 0, \xi)$, which implies $\limsup_{t\to\infty} k_t \leq k_{0,\xi}$. Let ξ converges to 0, we have that $k_{0,\xi}$ converges to k_x^*, and hence $\limsup_{t\to\infty} k_t \leq k_x^*$.

According to Lemma 4, we have $\lim_{t\to\infty} b_t = 0$. By combining this with condition $\limsup_{t\to\infty} k_t \leq k_x^*$, it is easy to prove that $\limsup_{t\to\infty} \frac{n^T}{\prod_{t=1}^T f'(k_t)} b_T = 0$. So, there is no bubble.

Part 2. Consider now the case $f'(k_x^*) < n$.

CASE 1: Consider the case $\liminf_{t\to\infty} k_t < x_n$.

If $\limsup_{t\to\infty} k_t < x_n$, there exists a sufficiently large T such that $k_{T+t} < x_n$ for any $t \geq 0$. It is possible to show that the solution is bubbleless using the same arguments of the case of a selfish economy. Point (iii) of Lemma 3 implies that the equilibrium is unique.

Assume now that $\limsup_{t\to\infty} k_t \geq x_n$. Suppose that the solution is bubbly. According to point (iii) of Proposition 1, we have $\lim_{t\to\infty} \xi_t/b_t = 0$.

Let $\bar{x}$ satisfy $\liminf_{t\to\infty} k_t < \bar{x} < x_n$. Since $\limsup_{t\to\infty} k_t \geq x_n$, there exists T high enough satisfying $k_{T+1} \leq k_T$, $k_{T+1} \leq \bar{x}$ and $\xi_{T+t}/b_{T+t} \leq f'(\bar{x})/n - 1$ for any $t \geq 0$. For this T, we have

$$b_{T+1} = \frac{f'(k_{T+1})}{n} b_T - \xi_{T+1} \geq \frac{f'(\bar{x})}{n} b_T - \xi_T \geq b_T$$

and, therefore, $k_{T+2} = \mathcal{G}_x(k_{T+1}, b_{T+1}, \xi_{T+1}) \leq \mathcal{G}_x(k_T, b_T, \xi_T) = k_{T+1} < \bar{x}$. By induction, the sequence $(k_{T+t})_{t=0}^{\infty}$ is decreasing and converges to some value which is smaller that $\bar{x} < x_n$. This leads to a contradiction with the hypothesis $\limsup_{t\to\infty} k_t \geq x_n$. Hence, the solution is bubbleless.

CASE 2: $\liminf_{t\to\infty} k_t \geq x_n$.

CASE 2.1. Focus first on the case $\liminf_{t\to\infty} k_t > x_n$. There exist $\epsilon > 0$ small and T high enough such that, for any $t \geq T$, we have $k_t > x_n + \epsilon$. This implies

$$b_{t+1} < \frac{f'(k_{t+1})}{n} b_t < \frac{f'(x_n + \epsilon)}{n} b_t$$

Thus, the sequence (b_t) is decreasing and converges to 0.

Fix $b > 0$ and $\xi > 0$. Take T sufficiently high such that $b_{T+t} < b$, $\xi_{T+t} < \xi$ for any $t \geq 0$. Then, $\mathcal{G}_x^t(k_T, b, 0) \leq k_{T+t} \leq \mathcal{G}_x^t(k_T, 0, \xi)$ and, for any $b > 0$ and $\xi > 0$, $\liminf_{t\to\infty} k_t \geq k_{b,0}$ and $\limsup_{t\to\infty} k_t \leq k_{0,\xi}$.

Let b, ξ tend to 0 we get $\liminf_{t\to\infty} k_t = \limsup_{t\to\infty} k_t = k_x^*$.

CASE 2.2. Consider now the case $\liminf_{t\to\infty} k_t = x_n$. First, we prove that $\liminf_{t\to\infty} b_t \geq b_n$ where b_n satisfies $x_n = \mathcal{G}_x(x_n, b_n, 0)$. Suppose the contrary. Fix b such that $\liminf_{t\to\infty} b_t < b < b_n$. From $\mathcal{G}_x(k_{b,0}, b, 0) = k_{b,0}$ and $x_n < \mathcal{G}_x(x_n, b, 0)$, we get $k_{b,0} > x_n$. Since $\mathcal{G}_x(x_n, b, 0) > x_n$, we can take $\epsilon > 0$ satisfying $\mathcal{G}_x(x_n - \epsilon, b, 0) > x_n + \epsilon$. Take also T high enough such that $k_T > x_n - \epsilon$ and $b_T < b$. We find

$$k_{T+1} = \mathcal{G}_x(k_T, b_T, \xi_T) \geq \mathcal{G}_x(k_T, b, 0) > x_n + \epsilon$$
$$b_{T+1} = \frac{f'(k_{T+1})}{n} b_T - \xi_{T+1} < b_T < b$$

By induction, we obtain $k_{T+t} > x_n + \epsilon$ and $b_{T+t} < b$ for any t. Hence, $\liminf_{t\to\infty} k_t \geq x_n + \epsilon > x_n$, that is a contradiction.

Since $\liminf_{t\to\infty} b_t \geq b_n$ for any $b < b_n$ and $\xi > 0$, there exists T satisfying $b_{T+t} > b$ and $\xi_{T+t} < \xi$ for any t. This implies $k_{T+t} < \mathcal{G}_x^t(k_T, b, \xi)$ and $\limsup_{t\to\infty} k_t \leq \mathcal{G}_x^t(k_T, b, \xi) = k_{b,\xi}$. Let b converge to b_n. ξ converges to 0. Thus, $k_{b,\xi}$ converges to $k_{b_n,0} = x_n$ and $\limsup_{t\to\infty} k_t \leq x_n \leq \liminf_{t\to\infty} k_t$. Hence, $\lim_{t\to\infty} k_t = x_n$ and $\lim_{t\to\infty} b_t = b_n$. □

Remark 11. *In the proof of cases* (2.b) *and* (2.c) *of Proposition 2, we do not use the monotonicity of* $(\xi_t)_{t=0}^{\infty}$.

11 Appendix: proofs of Section 6

Proof of Proposition 3. Part 1. Since k_t is bounded from above, according to Corollary 1, if $\limsup_{t\to\infty} \xi_t > 0$, there is no bubble.

Consider the case $\lim_{t\to\infty}\xi_t = 0$. Let us prove that $\lim_{t\to\infty} b_t/(Ak_t^\alpha) = 0$. If the contrary holds, $\limsup_{t\to\infty} b_t/(Ak_t^\alpha) > 0$. Since $\xi_t \to 0$, we can choose $\varepsilon > 0$ small and T high enough such that $\sigma b_T/(Ak_T^\alpha) > 2\varepsilon\alpha\gamma_x$ and $(1-\sigma)\xi_t/(Ak_t^\alpha) < \varepsilon\alpha\gamma_x$ for any $t \geq T$ and $\varepsilon\alpha\gamma_x (1/[(1-\varepsilon)\gamma_x] - 1) > \xi_{t+1}/(Ak_{t+1}^\alpha)$ for any $t \geq T$. Therefore, we obtain

$$\begin{aligned}
\frac{b_{T+1}}{Ak_{T+1}^\alpha} - \frac{b_T}{Ak_T^\alpha} &= \frac{b_T}{Ak_T^\alpha}\left[\frac{\alpha}{\alpha A\gamma_x + (1-\sigma)\xi_T/(Ak_T^\alpha) - \sigma b_T/(Ak_T^\alpha)} - 1\right] - \frac{\xi_{T+1}}{Ak_{T+1}^\alpha} \\
&> \frac{b_T}{Ak_T^\alpha}\left(\frac{\alpha}{\alpha\gamma_x + \varepsilon\alpha\gamma_x - 2\varepsilon\alpha\gamma_x} - 1\right) - \frac{\xi_{T+1}}{Ak_{T+1}^\alpha} \\
&= \frac{b_T}{Ak_T^\alpha}\left[\frac{1}{(1-\varepsilon)\gamma_x} - 1\right] - \frac{\xi_{T+1}}{Ak_{T+1}^\alpha} > 0
\end{aligned}$$

By induction, we find that the sequence $(b_t/(Ak_t^\alpha))_{t=T}^\infty$ is increasing and, hence, it converges to $\theta > 0$. The limit θ solves the equation $\theta = \theta\alpha/(\alpha\gamma_x - \sigma\theta)$. But, this is impossible under the assumption $\gamma_x \leq 1$. Thus, $\lim_{t\to\infty} b_t/(Ak_t^\alpha) = 0$: the relative size of asset value asymptotically vanishes.

Part 2. Consider an equilibrium (b_t, k_{t+1}). Conditions (14) and (15) give

$$\frac{b_{t+1}}{Ak_{t+1}^\alpha} = \frac{\alpha b_t}{nk_{t+1}} - \frac{\xi_{t+1}}{Ak_{t+1}^\alpha} = \frac{b_t}{Ak_t^\alpha}\frac{\alpha}{\alpha\gamma_x + (1-\sigma)\xi_t/(Ak_t^\alpha) - \sigma b_t/(Ak_t^\alpha)} - \frac{\xi_{t+1}}{Ak_{t+1}^\alpha}$$

(1) Focus on the first case: there exists T such that $b_T/(Ak_T^\alpha) \leq \theta_x^*$. Then,

$$\frac{b_{t+1}}{Ak_{t+1}^\alpha} < \frac{b_t}{Ak_t^\alpha}\frac{\alpha}{\alpha\gamma_x + (1-\sigma)\xi_t/(Ak_t^\alpha) - \sigma b_t/(Ak_t^\alpha)} < \frac{b_t}{Ak_t^\alpha}\frac{\alpha}{\alpha\gamma_x - \sigma\theta_x^*} = \frac{b_t}{Ak_t^\alpha} < \theta_x^*$$

for any $t \geq T$. The sequence $(b_t/(Ak_t^\alpha))$ is decreasing. This implies the existence of $\lim_{t\to\infty} b_t/(Ak_t^\alpha) \equiv \theta$, with $0 \leq \theta < \theta_x^*$. Let us show that $\theta = 0$. Suppose that $\theta > 0$. θ becomes solution of $\theta = \alpha\theta/(\alpha\gamma_x - \sigma\theta)$ that is $\theta = \theta_x^*$: a contradiction. Thus, we have $\lim_{t\to\infty} b_t/(Ak_t^\alpha) = 0$.

(2) Focus on the second case: we have $b_t/(Ak_t^\alpha) > \theta_x^*$ for every t. Let us prove that $\lim_{t\to\infty} b_t/(Ak_t^\alpha) = \theta_x^*$. If the contrary holds, $\limsup_{t\to\infty} b_t/(Ak_t^\alpha) = \theta > \theta_x^*$ which implies in turn the existence of $\varepsilon > 0$ and T high enough such that $b_T/(Ak_T^\alpha) > (1+\varepsilon)\theta_x^*$. Since $\xi_t \to 0$, we observe that

$$\begin{aligned}
&\lim_{t\to\infty}(1+\varepsilon)\theta_x^*\frac{\alpha}{\alpha\gamma_x + (1-\sigma)\xi_t/(Ak_t^\alpha) - \sigma(1+\varepsilon)\theta_x^*} - (1+\varepsilon)\theta_x^* \\
= \;&(1+\varepsilon)\theta_x^*\frac{\alpha}{\alpha\gamma_x - \sigma(1+\varepsilon)\theta_x^*} - (1+\varepsilon)\theta_x^* > 0
\end{aligned}$$

Thus, there exists T high enough such that $b_T/(Ak_T^\alpha) > (1+\varepsilon)\theta_x^*$ and, for every $t \geq T$,

$$\frac{\alpha(1+\varepsilon)\theta_x^*}{\alpha\gamma_x + (1-\sigma)\xi_t/(Ak_t^\alpha) - \sigma(1+\varepsilon)\theta_x^*} - (1+\varepsilon)\theta_x^* - \frac{\xi_{t+1}}{A(\inf_s k_s)^\alpha} > 0$$

Therefore,

$$\begin{aligned}
\frac{b_{T+1}}{Ak_{T+1}^\alpha} &= \frac{b_T}{Ak_T^\alpha}\frac{\alpha}{\alpha\gamma_x + (1-\sigma)\xi_T/(Ak_T^\alpha) - \sigma b_T/(Ak_T^\alpha)} - \frac{\xi_{T+1}}{Ak_{T+1}^\alpha} \\
&> \frac{\alpha(1+\varepsilon)\theta_x^*}{\alpha\gamma_x + (1-\sigma)\xi_T/(Ak_T^\alpha) - \sigma(1+\varepsilon)\theta_x^*} - \frac{\xi_{T+1}}{A(\inf_t k_t)^\alpha} \\
&> (1+\varepsilon)\theta_x^*
\end{aligned}$$

By induction, we find, for every $t \geq T$, $b_t/(Ak_t^\alpha) > (1+\varepsilon)\,\theta_x^*$ and

$$\begin{aligned}\frac{b_{t+1}}{Ak_{t+1}^\alpha} - \frac{b_t}{Ak_t^\alpha} &= \frac{b_t}{Ak_t^\alpha}\frac{\alpha - \alpha\gamma_x - (1-\sigma)\,\xi_t/(Ak_t^\alpha) + \sigma b_t/(Ak_t^\alpha)}{\alpha\gamma_x + (1-\sigma)\,\xi_t/(Ak_t^\alpha) - \sigma b_t/(Ak_t^\alpha)} - \frac{\xi_{t+1}}{Ak_{t+1}^\alpha} \\ &> (1+\varepsilon)\,\theta_x^* \frac{\alpha - \alpha\gamma_x - (1-\sigma)\,\xi_t/(Ak_t^\alpha) + \sigma(1+\varepsilon)\,\theta_x^*}{\alpha\gamma_x + \xi_t/(Ak_t^\alpha) - \sigma(1+\varepsilon)\,\theta_x^*} - \frac{\xi_{t+1}}{Ak_{t+1}^\alpha} \\ &= (1+\varepsilon)\,\theta_x^* \frac{\sigma\varepsilon\theta_x^* - (1-\sigma)\,\xi_t/(Ak_t^\alpha)}{\alpha\gamma_x + \xi_t/(Ak_t^\alpha) - \sigma(1+\varepsilon)\,\theta_x^*} - \frac{\xi_{t+1}}{Ak_{t+1}^\alpha}\end{aligned}$$

This implies that $\liminf_{t\to\infty}\left[b_{t+1}/(Ak_{t+1}^\alpha) - b_t/(Ak_t^\alpha)\right] > 0$: for T high enough, the sequence $(b_t/(Ak_t^\alpha))_{t=T}^\infty$ is increasing and converges to $\theta > \theta_x^*$. Applying the same argument of point (1), we get $\theta = \theta_x^*$, that is a contradiction.

It is immediate to see that $\lim_{t\to\infty} b_t/(Ak_t^\alpha) = \theta_x^*$ and, then, $k_t \to x_n$, and $b_t \to n\,(\gamma_x - 1)\,x_n/\sigma$ when t tends to infinity. □

Proof of Example 1. The equilibrium system (14)-(15) writes

$$k_{t+1} = \frac{\alpha A\gamma_0 k_t^\alpha - b_t}{n}, \quad b_{t+1} = \frac{\alpha A b_t}{nk_{t+1}^{1-\alpha}} - \xi_{t+1}. \tag{35}$$

The proof is articulated in two steps.

STEP 1. Let (x_t) be a positive sequence such that

$$x_t \geq \frac{1}{\gamma_0} \text{ and } x_t + \frac{1}{\gamma_0 x_{t+1}} \geq \frac{1}{\gamma_0} + 1 \tag{36}$$

for every t. We prove that there exists a sequence of nonnegative dividends (ξ_t) such that (b_t, k_{t+1}) is solution of system (14)-(15) with

$$k_{t+1} = \frac{\alpha A k_t^\alpha}{n x_t} \forall t. \tag{37}$$

To show that such an asset exists, consider the sequence (b_t, k_{t+1}) defined by (37) and $b_t = \alpha A\gamma_0 k_t^\alpha - nk_{t+1}$. Since $x_t \geq 1/\gamma_0$, we have $b_t = \alpha A\gamma_0 k_t^\alpha - nk_{t+1} \geq 0$ for every t. We define the sequence (ξ_t) with dividends (16) for every $t \geq 0$. Then,

$$\begin{aligned}\xi_{t+1} &= \frac{\alpha A\left(\alpha A\gamma_0 k_t^\alpha - nk_{t+1}\right)}{nk_{t+1}^{1-\alpha}} - \left(\alpha A\gamma_0 k_{t+1}^\alpha - nk_{t+2}\right) \\ &= \alpha A\gamma_0 k_{t+1}^\alpha \left(\frac{\alpha A k_t^\alpha}{nk_{t+1}} - \frac{1}{\gamma_0} - 1 + \frac{nk_{t+2}}{\alpha A\gamma_0 k_{t+1}^\alpha}\right) \\ &= \alpha A\gamma_0 k_{t+1}^\alpha \left(x_t + \frac{1}{\gamma_0 x_{t+1}} - 1 - \frac{1}{\gamma_0}\right)\end{aligned} \tag{38}$$

According to inequality (36), we see that $\xi_{t+1} \geq 0$ for every $t \geq 0$ and, therefore, (b_t, k_{t+1}) is solution of system (35) with sequence of dividends (ξ_t).

STEP 2. Let us now prove Example 1. We have

$$x_t = e^{\lambda^t} \tag{39}$$

for every $t > 0$, and $x_0 \equiv \max\{e, 1 + 2/\gamma_0\} \geq e$. The sequence (x_t) satisfies restrictions (36). Consider the sequence (b_t, k_{t+1}) defined by (37) and $b_t = \alpha A\gamma_0 k_t^\alpha - nk_{t+1}$ jointly with the sequence of dividends (16). We have $\lim_{t\to\infty} x_t = \infty$ and, according to (38), $\xi_{t+1} > 0$ for every t. Thus, the sequence (b_t, k_{t+1}) is solution of system (35) with $\lim_{t\to\infty} k_t = \lim_{t\to\infty} b_t = 0$.

Let us prove now that $\lim_{t\to\infty} \xi_{t+1} = 0$ or, equivalently, that $\lim_{t\to\infty} k_{t+1}^{\alpha} x_t = 0$. Solving recursively (37), we find

$$k_{t+1}^{\alpha} = \left(\frac{\alpha A}{n}\right)^{\alpha\frac{1-\alpha^{t+1}}{1-\alpha}} \frac{k_0^{\alpha^{t+2}}}{x_0^{\alpha^{t+1}}} \Pi_{s=0}^{t-1} \frac{1}{x_{t-s}^{\alpha^{1+s}}}$$

and, using (39),

$$k_{t+1}^{\alpha} = \left(\frac{\alpha A}{n}\right)^{\alpha\frac{1-\alpha^{t+1}}{1-\alpha}} \frac{k_0^{\alpha^{t+2}}}{x_0^{\alpha^{t+1}}} e^{-\sum_{s=0}^{t-1}\alpha^{1+s}\lambda^{t-s}}$$

We notice that λ is solution of $\lambda^t = \sum_{s=0}^{2} \alpha^{1+s}\lambda^{t-s}$. Then, for $t > 4$,

$$\begin{aligned} k_{t+1}^{\alpha} x_t &< \left(\frac{\alpha A}{n}\right)^{\alpha\frac{1-\alpha^{t+1}}{1-\alpha}} \frac{k_0^{\alpha^{t+2}}}{x_0^{\alpha^{t+1}}} e^{-\sum_{s=0}^{3}\alpha^{1+s}\lambda^{t-s}} x_t \\ &= \left(\frac{\alpha A}{n}\right)^{\alpha\frac{1-\alpha^{t+1}}{1-\alpha}} \frac{k_0^{\alpha^{t+2}}}{x_0^{\alpha^{t+1}}} e^{\lambda^t-\sum_{s=0}^{3}\alpha^{1+s}\lambda^{t-s}} = \left(\frac{\alpha A}{n}\right)^{\alpha\frac{1-\alpha^{t+1}}{1-\alpha}} \frac{k_0^{\alpha^{t+2}}}{x_0^{\alpha^{t+1}}} e^{-\alpha^4\lambda^{t-3}} \end{aligned}$$

and, in the limit, we get $\lim_{t\to\infty} k_{t+1}^{\alpha} x_t \le 0$. □

Proof of Example 2. Let (b_t, k_{t+1}) be a sequence determined by the system (14)-(15) and $b_0 \in (\theta_x^* A k_0^{\alpha}/n, \theta_x^* A k_0^{\alpha})$. To prove that this is an equilibrium, we check that $b_t > 0$ and $k_{t+1} > 0$ for any t.

According to (14), we see that $k_{t+1} > 0$ if $\alpha A \gamma_x k_t^{\alpha} > \sigma b_t$, which is satisfied when $b_t < \theta_x^* A k_t^{\alpha}$. Therefore, we have just ot prove that $b_t \in (\theta_x^* A k_t^{\alpha}/n^{t+1}, \theta_x^* A k_t^{\alpha})$ for every t. Let us apply the induction argument.

(1) We show first that $b_t < \theta_x^* A k_t^{\alpha}$ implies $b_{t+1} < \theta_x^* A k_{t+1}^{\alpha}$. Indeed, considering (14) and (15), we find

$$\begin{aligned} \frac{b_{t+1}}{Ak_{t+1}^{\alpha}} &= \frac{b_t\alpha}{Ak_t^{\alpha}\alpha\gamma_x + (1-\sigma)\xi_t - \sigma b_t} - \frac{\xi_{t+1}}{Ak_{t+1}^{\alpha}} = \frac{b_t}{Ak_t^{\alpha}} \frac{\alpha}{\alpha\gamma_x + \frac{(1-\sigma)\xi_t - \sigma b_t}{Ak_t^{\alpha}}} - \frac{\xi_{t+1}}{Ak_{t+1}^{\alpha}} \\ &\le \frac{b_t}{Ak_t^{\alpha}} \frac{\alpha}{\alpha\gamma_x - \frac{\sigma b_t}{Ak_t^{\alpha}}} - \frac{\xi_{t+1}}{Ak_{t+1}^{\alpha}} < \frac{b_t}{Ak_t^{\alpha}} \frac{\alpha}{\alpha\gamma_x - \sigma\theta_x^*} = \frac{b_t}{Ak_t^{\alpha}} \le \theta_x^* \end{aligned}$$

(2) Then, we prove that $b_t > \theta_x^* A k_t^{\alpha}/n^{t+1}$ implies $b_{t+1} > \theta_x^* A k_{t+1}^{\alpha}/n^{t+2}$. Since $b_t \le \theta_x^* A k_t^{\alpha}$ for every t, we have $k_t \ge k_m$ for every t. Using (14) and (15), we obtain

$$\begin{aligned} \frac{b_{t+1}}{Ak_{t+1}^{\alpha}} - \frac{\theta_x^*}{n^{t+2}} &= \frac{b_t}{Ak_t^{\alpha}} \frac{\alpha}{\alpha\gamma_x + \frac{(1-\sigma)\xi_t - \sigma b_t}{Ak_t^{\alpha}}} - \frac{\xi}{n^{t+1}Ak_{t+1}^{\alpha}} - \frac{\theta_x^*}{n^{t+2}} \\ &> \frac{\theta_x^*}{n^{t+1}} \frac{\alpha}{\alpha\gamma_x + \frac{(1-\sigma)\xi}{n^t Ak_t^{\alpha}} - \frac{\sigma\theta_x^*}{n^{t+1}}} - \frac{\xi}{n^{t+1}Ak_{t+1}^{\alpha}} - \frac{\theta_x^*}{n^{t+2}} \\ &> \frac{\theta_x^*}{n^{t+1}} \left[\frac{\alpha}{\alpha\gamma_x + \frac{(1-\sigma)\xi}{n^t Ak_t^{\alpha}}} - \frac{1}{n} - \frac{\xi}{\theta_x^* A k_{t+1}^{\alpha}}\right] \\ &\ge \frac{\theta_x^*}{n^{t+1}} \left[\frac{\alpha}{\alpha\gamma_x + \frac{(1-\sigma)\xi}{Ak_m^{\alpha}}} - \frac{1}{n} - \frac{\xi}{\theta_x^* A k_m^{\alpha}}\right] > 0 \end{aligned}$$

□

Proof of Proposition 4. A bubble exists if and only if $b_t > 0$ for any t.

Combining (18) and (19), we get a single dynamic equation:

$$z_{t+1} = \gamma_x z_t - 1 \quad \forall t \geq 0 \tag{40}$$

where $z_t \equiv nk_{t+1}/(\sigma b_t)$. The solution of the difference equation (40) is given by

$$z_t = \gamma_x^t z_0 - \frac{1-\gamma_x^t}{1-\gamma_x} \quad \forall t \geq 1$$

provided that $\gamma_x \neq 1$.

(1) When $\gamma_x \leq 1$, there is no bubble. Indeed, if $\gamma_x \leq 1$, z_t becomes negative soon or later: this leads to a contradiction. In this case, capital transition becomes $k_{t+1} = \rho_{\gamma_x} k_t^\alpha$. Solving recursively, we find the explicit solution (21). We observe that, according to (20), $\lim_{t\to\infty} k_t = \rho_{\gamma_x}^{1/(1-\alpha)} = k_x^*$.

(2) Let $\gamma_x > 1$.

(2.a) If $b_t = 0$, then (21) follows immediately.

(2.b) Focus on the case $b_t > 0$. Then, we obtain

$$z_t = \frac{[(\gamma_x - 1)\, z_0 - 1]\, \gamma_x^t + 1}{\gamma_x - 1} \tag{41}$$

A positive solution exists if and only if $z_0 \geq 1/(\gamma_x - 1)$. Hence, the existence of a positive solution requires

$$b_0 \leq \frac{\gamma_x - 1}{\sigma} nk_1 = \frac{\gamma_x - 1}{\sigma}\left[\frac{\beta}{1+\beta}(w_0 + g_0) - b_0\right]$$

Solving this inequality for b_0, we find $0 < b_0 \leq \bar{b}_x$.

Now, given $b_0 \in (0, \bar{b}_x]$, the sequence (k_{t+1}, b_t) constructed by (18) and (19) is an equilibrium with $b_t > 0$ for any t.

When $b_0 < \bar{b}_x$ (that is $z_0 > 1/(\gamma_x - 1)$), because of (41), we get $\lim_{t\to\infty} z_t = \infty$. According to (18), k_t is uniformly bounded from above, which implies that $\lim_{t\to\infty} b_t = 0$. Thus, $\lim_{t\to\infty} k_t = k_x^*$.

When $b_0 = \bar{b}_x$, we have $z_t = 1/(\gamma_x - 1)$ for any $t \geq 0$. In this case, $k_{t+1} = \rho_1 k_t^\alpha$ where $\rho_1 \equiv \alpha A/n$ for any $t > 0$ and $b_t = (\gamma_x - 1)\, nk_{t+1}/\sigma$. Solving recursively, we get the explicit solution (22). □

Proof of Proposition 5. (1) When $R > n$.

Denote $D \equiv \frac{R}{n}\frac{\beta}{1+\beta}\frac{x}{1+x}$. According to (26), and using the fact that $b_t + \xi_t = \frac{R}{n} b_{t-1}$ we have

$$k_{t+1} = Dk_t + \frac{1}{n}\frac{\beta}{1+\beta}\Big(w + \frac{x}{1+x}\xi_t\Big) - \frac{1}{n}\Big(1 - \frac{\beta}{1+\beta}\frac{x}{1+x}\Big)b_t \tag{42}$$

$$= Dk_t + \frac{1}{n}\frac{\beta}{1+\beta}w + \frac{1}{n}(Db_{t-1} - b_t). \tag{43}$$

So, we find that $nk_{t+1} + b_t = D(nk_t + b_{t-1}) + \frac{\beta w}{1+\beta}$, and therefore

$$\frac{nk_{t+1} + b_t}{D^t} = \frac{nk_t + b_{t-1}}{D^{t-1}} + \frac{\beta w}{1+\beta}\frac{1}{D^t}$$

$$\Longrightarrow \frac{nk_{t+1} + b_t}{D^t} = nk_1 + b_0 + \frac{\beta w}{1+\beta}\sum_{s=1}^{t}\frac{1}{D^s}$$

$$\Longrightarrow nk_{t+1} + b_t = D^t(nk_1 + b_0) + \frac{\beta w}{1+\beta}\sum_{s=0}^{t-1} D^s = D^t(nk_1 + b_0) + \frac{\beta w}{1+\beta}\frac{D^t - 1}{D - 1}. \tag{44}$$

Since $nk_1 + b_0 = \frac{\beta}{1+\beta}(w + g_0)$, we obtain

$$nk_{t+1} = D^t(nk_1 + b_0) + \frac{\beta w}{1+\beta}\frac{D^t - 1}{D - 1} - \frac{R^t}{n^t}\left(b_0 - \sum_{s=1}^{t} \frac{n^s}{R^s}\xi_s\right).$$

If bubbles exist, then, $b_0 > \sum_{s=1}^{\infty} \frac{n^s}{R^s}\xi_s$. Let us denote $B := b_0 - \sum_{s=1}^{\infty} \frac{n^s}{R^s}\xi_s$. Since $D < R/n$, we observe that $b_t > \left(\frac{R}{n}\right)^t B$, which converges to infinity and grows faster than the right hand side of (44. Hence, k_{t+1} will be strictly negative for t high enough, a contradiction. Hence, bubbles are ruled out.

(2) When $R \leq n$. The proof in this case is easy. □

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