A Dynamic Programming Approach in Which Consuming Takes Time

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Abstract

This article establishes a growth model in which consumers face a linear time constraint. By using a dynamic programming argument, it is proved that the optimal capital sequences are monotonic and have property that converges to steady state.

Keywords. allocation of time, dynamic programming, value function.

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1 Introduction

The standard economic models based on assumptions either that consumption is instantaneous or that a economic agent has a sufficiently large amount of time to consume any finite bundle of commodities. Neither of these assumptions is plausible. Consumption takes time and every agent has only a finite amount of time to allocate between working, consumption and leisure. Note further that the act of consumption is broadly interpreted to include search, purchase, preparation, and consumption.

The consumption time constraint was first formally introduced by Gossen [7] and much later generalized by Becker [1] in the famous theory of time allocation. Binh Tran-Nam and Sang Pham-Ngoc [2] introduced a consumption technological coefficient $a > 0$ to set up the linear time constraint in a simplified model with representative households. Cuong Le-Van and al. [3] extended the general equilibrium model with multi factors and many heterogeneous economic agents.

In this article, we establish a growth model in which consumption is itself time consuming. The model also produces results which similar to those obtained from a single-sector growth model with elastic labor which discussed in [6]. The key advance of this article is to prove the monotonicity of the optimal capital sequences and to prove the existence of optimal steady state under supermodularity of technology with dynamic programming tools.

The remainder of this paper is organized as follows. Section 2 describes the model. Section 3 characterizes the indirect utility function. Section 4 is devoted to the study of the value function. Section 5 deals with the properties of optimal paths. Some long proofs are gathered in the Appendix.

2 The Model

We consider an economy populated by consumers which are identical in all respects. Preferences are represented by the function

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

where u is the instantaneous utility function and $\beta \in (0, 1)$ is the discount factor.

The capital stock and consumption of consumer at period t are denoted by k_t and c_t respectively. We assume that the initial capital stock k_0 and labour endowment L (measured in time units) are exogenous.

In this economy there is one good produced by a single firm using as inputs physical capital (k_t) and labour (l_t).

$$Y_t = F(k_t, l_t)$$

The dynamics of the sequence of capital stocks:

$$k_{t+1} = (1-\delta)k_t + I_t$$

where I_t is investment and $\delta \in [0,1]$ is the rate of depreciation of capital.

In each period , consumers face resource constraint:

$$c_t + I_t \leq Y_t$$

Denote $a > 0$ is the technological coefficient associated with this good. At period t, the consumer purchases the quantity c_t of good and spends ac_t of time to transform these consumption goods to the "final" consumption goods that the consumer wishes to enjoy. The consumer also spends l_t of time to work in firm. The time constraint is given by

$$ac_t + L_t \leq L$$

The social planner wants to maximize the global utility of the consumer:

$$(P) \qquad \max_{(c_t, k_{t+1}, l_t)} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

under the constraints:

$$c_t + k_{t+1} - (1-\delta)k_t \leq F(k_t, l_t) \quad (1)$$

$$ac_t + l_t \leq L \quad (2)$$

$$c_t \geq 0, l_t \geq 0, k_{t+1} \geq 0 \quad (3)$$

$$k_0 \geq 0 \text{ is given}$$

We next specify the properties assumed for the preferences and the technology.

The Assumptions

Assumption P1. $k_0 \geq 0, L > 0, a > 0$

Assumption P2. *The utility function $u(c)$ is continuous, strictly increasing, strictly concave, $u(0) = 0$.*

Assumption P3. *u is continuous differentiable on $\mathbb{R}_+$, $\lim_{c \to 0} u_c(c) = +\infty$.*

Assumption P4. *The production function $F(k, l)$ is concave, strictly increasing for $k > 0, l > 0$, $F(k, 0) = F(0, l) = 0$.*

Assumption P5. *$F(k,l)$ is twice continuously differentiable on $\mathbb{R}^2_{++}$. In addition,* $\lim_{k\to+\infty} F_k(k,L) = 0, \lim_{l\to 0} F_l(k,L) = +\infty$.

Assumption P6. $\lim_{k\to 0} F_k(k,L) > \frac{1}{\beta} - 1 + \delta$.

Remark 2.1. *In growth model with elastic labor, the social planning problem is determining a consumption-leisure allocation and production sequence:* [1]

$$(Q) \qquad \max_{(c_t,k_{t+1},\mathfrak{L}_t)} \sum_{t=0}^{\infty} \beta^t u^Q(c_t,\mathfrak{L}_t)$$

under the constraints:

$$c_t + k_{t+1} - (1-\delta)k_t \leq F(k_t,l_t) \tag{4}$$

$$\mathfrak{L}_t + l_t \leq L \tag{5}$$

$$c_t \geq 0, l_t \geq 0, \mathfrak{L} \geq 0, k_{t+1} \geq 0 \tag{6}$$

$$k_0 \geq 0 \text{ is given}$$

where $\mathfrak{L}_t$ denotes the quantity of leisure which the consumer spends in period t.

In addition, the assumptions used for the problem (Q) are similar with those of problem (P).

3 Characterization of the Indirect Utility Function

Consider the problem (P). The two following lemmas claim that the budget constraint and the time constraint are binding at optimal.

Lemma 3.1. *Assume* **P1** *and* **P2***. If $(c_t^*, k_{t+1}^*, l_t^*)$ is an optimal solution of (P) then, for any $t \geq 0$:*

$$ac_t^* + l_t^* = L. \tag{7}$$

Proof. If $ac_t^* + l_t^* < L$, we can increase both c_t^* and l_t^* such that

$$a(c_t^* + \epsilon_c) + (l_t^* + \epsilon_l) \leq L.$$

and

$$(c_t^* + \epsilon_c) + k_{t+1}^* - (1-\delta)k_t^* \leq F(k_t^*, l_t^* + \epsilon_l).$$

In this case we have $u(c_t^* + \epsilon_c) > u(c_t^*)$ contradiction with the optimal solution of (c_t^*, l_t^*). □

[1] For the problem Q, we use the results from the working paper [6] of Cuong Le Van and Yiannis Vailakis entitle "Existence of Competitive Equilibrium in a Single-Sector Growth Model with Elastic Labor"

Then the problem (P) is equivalent to

$$(P') \qquad \max_{(k_{t+1}, l_t)} \sum_{t=0}^{\infty} \beta^t u\left(\frac{L - l_t}{a}\right)$$

under the constraints: $(\forall t \geq 0)$

$$0 \leq k_{t+1} \leq F(k_t, l_t) + (1-\delta)k_t + \frac{l_t - L}{a}$$

$$0 \leq l_t \leq L$$

$$k_0 \geq 0 \text{ is given}$$

Lemma 3.2. *Assume* **P1** *and* **P2**. *If* $(c_t^*, k_{t+1}^*, l_t^*)$ *is an optimal solution of* (P) *then, for any* $t \geq 0$*:*

$$l_t^* > 0 \tag{8}$$

$$k_{t+1}^* = F(k_t^*, l_t^*) + (1-\delta)k_t^* + \frac{l_t^* - L}{a} \tag{9}$$

$$(1-\delta)k_t^* - \frac{L}{a} \leq k_{t+1}^* \leq F(k_t^*, L) + (1-\delta)k_t^* \tag{10}$$

Proof. See appendix. □

To make further analysis we will study the value function. Working in this direction, we define the technology set as following:

Let

$$\Gamma(k) = \{k' : \max\{0, (1-\delta)k - \frac{L}{a}\} \leq k' \leq F(k, L) + (1-\delta)k\}$$

and

$$\text{Graph}\Gamma = \{(k, k') : k \geq 0, k' \in \Gamma(k)\}.$$

Given $k_0 \geq 0$, denote by $\Pi(k_0)$ the set of feasible capital sequences from k_0, i.e.

$$\Pi(k_0) = \{\mathbf{k} = (k_0, k_1, \ldots, k_t, \ldots) : k_{t+1} \in \Gamma(k_t) \ \forall t \geq 0\}.$$

Lemma 3.3. *Under assumptions* **P4, P5** *and* **P6**,

i) There exist $\bar{k} \geq 0$ *such that for any* $k' \in \Gamma(k)$ *then* $k' \leq \max(\bar{k}, k)$.

ii) The correspondence $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous, with non-empty, compact convex values. Moreover* $0 \in \Gamma(0)$.

iii) The set $\Pi(k_0)$ *is compact for the product topology.*

Proof. See appendix.

□

Definition 3.1. *For* $(k,y) \in Graph(\Gamma)$ *we define the indirect utility function* U *by:*

$$
\begin{aligned}
U(k,y) = \max_{(c,l)} u(c)& \\
s.t.\quad & c + y \leq F(k,l) + (1-\delta)k \\
& ac + l \leq L \\
& c \geq 0; \quad 0 \leq l \leq L
\end{aligned}
$$

The following proposition establishes the properties of the indirect utility function.

Proposition 3.1. *Assume* **P1-P6**.

1. *Assume* $y \in int(\Gamma(k))$. *Let* $c^*(k,y)$ *and* $l^*(k,y)$ *denote the solution to the problem defined in 3.1. We have* (c^*, l^*) *is unique with* $c^* > 0$ *and* $l^* \in (0, L)$.
2. U *is differentiable and*

$$
\begin{aligned}
\frac{\partial U(k,y)}{\partial k} &= \lambda[F'_k(k,l^*) + (1-\delta)] \\
\frac{\partial U(k,y)}{\partial y} &= -\lambda
\end{aligned}
$$

   where $\lambda = \dfrac{u'_c(c^*)}{1 + aF'_l(k,l^*)}$.

3. U *is increasing in its first variable and decreasing in its second one.*
4. U *is strictly concave. Moreover, there exist* $A \geq 0, B \geq 0$ *such that* $U(k,y) \leq A + B(k+y)$.

Proof. See appendix.

□

Corollary 3.1. *Assume* **P1-P6**. *For each* $(k,y) \in Graph\Gamma$, *we can rewrite the function* U *as following:*

$$U(k,y) = u\left(\frac{L-l}{a}\right)$$

with:

$$y = F(k,l) + (1-\delta)k + \frac{l-L}{a}.$$

Corollary 3.2. *The problem* (P) *is equivalent to:*

$$(P_2) \quad \max_{\Pi(k_0)} \sum_{t=0}^{\infty} \beta^t U(k_t, k_{t+1}), \quad k_0 \geq 0 \text{ is given}$$

Corollary 3.3. *Assume* **P1-P6**. *For every* $k_0 > 0$, *the sum* $\sum_{t=0}^{+\infty} \beta^t U(k_t, k_{t+1})$ *exists and is finite-valued.*

Proof. From proposition 3.1 and lemma 3.3, there exists $A \geq 0, B \geq 0, K \geq 0$:

$$\begin{aligned} U(k_t, k_{t+1}) &\leq A + B(k_t + k_{t+1}), \quad \forall t \geq 0 \\ &\leq A + 2BK \\ \Rightarrow \sum_{t=0}^{\infty} \beta^t U(k_t, k_{t+1}) &\leq (A + 2BK) \sum_{t=0}^{\infty} \beta^t < +\infty \end{aligned}$$

□

Remark 3.1 (Growth model with elastic labor - Continued)**.** *For the problem (Q), we define the correspondence:*

$$\Gamma^Q(k) = \{y \in \mathbb{R}_+ : 0 \leq y \leq (1-\delta)k + F(k, L)\}$$

and the technology set

$$Graph(\Gamma^Q) = \{(k, y) \in \mathbb{R}_+^2 : y \in \Gamma^Q(k)\}$$

Let $(k, y) \in Graph(\Gamma^Q)$, define the indirect utility function U^Q by:

$$\begin{aligned} U^Q(k, y) = \max u^Q(c, L - l) \\ s.t.\ c + y \leq (1-\delta)k + F(k, l) \\ c \geq 0, 0 \leq l \leq L \end{aligned}$$

The results are similar with those of problem (P):

- *U^Q is continuous at any $(k, y) \in Graph(\Gamma^Q)$ with $k > 0$.*
- *$U^Q(k, y)$ is increasing in k, decreasing in y and strictly concave in (k, y).*
- *The set of feasible capital sequences:*

  $$\Pi^Q(k_0) = \{\mathbf{k} \in (\mathbb{R}_+)^\infty : k_{t+1} \in \Gamma^Q(k_t), \forall t \geq 0\}$$

  is compact for the product topology

4 Characterization of the Value Function

Definition 4.1. *For $\mathbf{k} \in \Pi(k_0)$, we denote*

$$v(\mathbf{k}) = \sum_{t=0}^{+\infty} \beta^t U(k_t, k_{t+1}).$$

Let

$$\begin{aligned} V(k_0) &= \max\{v(k) : \mathbf{k} \in \Pi(k_0)\} \\ &= \max_{\mathbf{k}\in\Pi(k_0)} \sum_{t=0}^{+\infty} \beta^t U(k_t, k_{t+1}) \end{aligned}$$

be the Value Function of the optimal growth problem (P).

Lemma 4.1. *Under assumptions* **P1-P6***,*
i) The function v is continuous,
ii) The function V is strictly concave, continuous.

Proof. i) Let $\{k^n\}$ be a sequence of $\Pi(k_0)$ converging to $k \in \Pi(k_0)$. From lemma 3.3, for all $\varepsilon > 0$ there exists $T_0 > 0$ for all $T \geq T_0$:

$$\sum_{t=T}^{+\infty} \beta^t U(k_t^n, k_{t+1}^n) < \varepsilon, \qquad \sum_{t=T}^{+\infty} \beta^t U(k_t, k_{t+1}) < \varepsilon$$

Fix T. Because of continuity of U, there exists $N_0 \geq 0$, for all $N \geq N_0$:

$$|U(k_t^n, k_{t+1}^n) - U(k_t, k_{t+1})| < \varepsilon \frac{1-\beta}{1-\beta^T}$$

Then we have, for any $\varepsilon > 0$, any $N \geq N_0$:

$$\begin{aligned} |v(k^n) - v(k)| &= \left| \sum_{t=0}^{+\infty} \beta^t U(k_t^n, k_{t+1}^n) - \sum_{t=0}^{+\infty} \beta^t U(k_t, k_{t+1}) \right| \\ &\leq \sum_{t=T}^{+\infty} \beta^t U(k_t^n, k_{t+1}^n) + \sum_{t=T}^{+\infty} \beta^t U(k_t, k_{t+1}) + \sum_{t=0}^{T-1} \beta^t \left| U(k_t^n, k_{t+1}^n) - U(k_t, k_{t+1}) \right| \\ &\leq 2\varepsilon + \sum_{t=0}^{T} \beta^t \varepsilon \frac{1-\beta}{1-\beta^T} = 3\varepsilon \end{aligned}$$

this proves the continuity of v.

ii) Given $k_0 \geq 0, k'_0 \geq 0, k_0 \neq k'_0$. There exist sequences $k = (k_0, k_1, \ldots), k' = (k'_0, k'_1, \ldots)$ such that

$$V(k_0) = \sum_{t=0}^{+\infty} \beta^t U(k_t, k_{t+1}); \quad V(k'_0) = \sum_{t=0}^{+\infty} \beta^t U(k'_t, k'_{t+1})$$

Let $\lambda \in [0,1]$. Denote $k^\lambda = \lambda k + (1-\lambda)k' = (\lambda k_t + (1-\lambda)k'_t)_t$. Since $k \in \Pi(k_0), k' \in \Pi(k'_0)$ we have for all $t \geq 0$:

$$\begin{aligned} (1-\delta)k_t - \frac{L}{a} \geq & k_{t+1} \geq F(k_t, L) + (1-\delta)k_t \\ (1-\delta)k'_t - \frac{L}{a} \geq & k'_{t+1} \geq F(k'_t, L) + (1-\delta)k'_t \end{aligned}$$

this implies

$$\lambda k_{t+1} + (1-\lambda)k'_{t+1} \geq (1-\delta)[\lambda k_t + (1-\lambda)k'_t] - \frac{L}{a}$$

and

$$\begin{aligned}\lambda k_{t+1} + (1-\lambda)k'_{t+1} &\leq \lambda F(k_t, L) + (1-\lambda)F(k'_t, L) + (1-\delta)[\lambda k_t + (1-\lambda)k'_t] \\ &\leq F(\lambda k_t + (1-\lambda)k'_t, L) + (1-\delta)[\lambda k_t + (1-\lambda)k'_t]\end{aligned}$$

hence $k^\lambda \in \Pi(\lambda k_0 + (1-\lambda)k'_0)$. We have:

$$\begin{aligned}V(\lambda k_0 + (1-\lambda)k'_0) &\geq \sum_{t=0}^{+\infty} \beta^t U(k^\lambda_t, k^\lambda_{t+1}) \\ &= \sum_{t=0}^{+\infty} \beta^t U(\lambda k_t + (1-\lambda)k'_t, \lambda k_{t+1} + (1-\lambda)k'_{t+1}) \\ &> \sum_{t=0}^{+\infty} \beta^t [\lambda U(k_t, k_{t+1}) + (1-\lambda)U(k'_t, k'_{t+1})] \\ &= \lambda \sum_{t=0}^{+\infty} \beta^t U(k_t, k_{t+1}) + (1-\lambda)\sum_{t=0}^{+\infty} \beta^t U(k'_t, k'_{t+1}) \\ &= \lambda V(k_0) + (1-\lambda)V(k'_0)\end{aligned}$$

We have proved the concavity of V. Since V is concave on $[0, +\infty)$, it is continuous in $(0, +\infty)$. We will show that V is continuous at 0.

It is easy to see that $\Pi(0) = 0$ and $V(0) = 0$. Let k_0 converge to zero, $(k^*_t)_{t=0}^{\infty} \in \Pi(k_0)$ and

$$V(k_0) = \sum_{t=0}^{\infty} U(k^*_t, k^*_{t+1})$$

For any $\varepsilon > 0$ and for T large enough we have

$$\sum_{t=T}^{\infty} U(k^*_t, k^*_{t+1}) < \frac{\varepsilon}{2}$$

On the other hand, since $U(k^*_t, .)$ is decreasing we have

$$\begin{aligned}U(k^*_t, k^*_{t+1}) &\leq U(k^*_t, 0) \\ &= \max\{u(\frac{L-l}{a}) \colon\ F(k^*_t, l) + (1-\delta)k^*_t + \frac{l-L}{a} \geq 0\} \\ &\leq u(F(k^*_t, L) + (1-\delta)k^*_t)\end{aligned}$$

Denote $\gamma(k) = F(k, L) + (1-\delta)k$ and reminder that $k^*_{t+1} \leq \gamma(k^*_t)$ we have:

$$\begin{aligned}U(k_0, k^*_1) &\leq u(\gamma(k_0)) \\ U(k^*_1, k^*_2) &\leq U(\gamma(k_0), k^*_2) \leq u(\gamma^2(k_0)) \\ U(k^*_t, k^*_{t+1}) &\leq u(\gamma^{t+1}(k_0)) \quad \forall t \geq 0\end{aligned}$$

Hence

$$V(k_0) \leq \sum_{t=0}^{T-1} u(\gamma^{t+1}(k_0)) + \sum_{t=T}^{\infty} U(k_t^*, k_{t+1}^*)$$

using the continuity of u and γ, these exists a neighborhood of 0 such that

$$|u(\gamma^{t+1}(k_0)) - u(\gamma^{t+1}(0))| < \frac{\varepsilon}{2T}, \forall t \leq T \Rightarrow \sum_{t=0}^{T-1} u(\gamma^{t+1}(k_0)) < \frac{\varepsilon}{2}$$

then

$$V(k_0) < \varepsilon$$

implies that V is continuous at 0.

□

Proposition 4.1. *Under assumptions* **P1-P6***, there exist a unique solution to the problem (P)*

Proof. The problem (P) is equivalent to the maximization of a continuous function $v(.)$ over a compact set $\Pi(k_0)$, and therefore it admits a solution. From lemma 3.1, the function U is strictly concave, and then implies that the solution is unique. □

Proposition 4.2. *Under assumptions* **P1-P6***, the value function V satisfies the Bellman equation:*

$$V(k_0) = \max_{k' \in \Gamma(k_0)} \{U(k_0, k') + \beta V(k')\}$$

Proof. Let $k^* = (k_0, k_1^*, k_2^*, \ldots, k_t^*, \ldots)$ satisfy

$$V(k_0) = \max_{k \in \Pi(k_0)} \sum_{t=0}^{+\infty} \beta^t U(k_t, k_{t+1}) = \sum_{t=0}^{+\infty} \beta^t U(k_t^*, k_{t+1}^*)$$

We have:

$$\begin{aligned} V(k_0) &= U(k_0, k_1^*) + \sum_{t=1}^{+\infty} \beta^t U(k_t^*, k_{t+1}^*) \\ &= U(k_0, k_1^*) + \beta v(k_1^*, k_2^*, \ldots, k_t^*, \ldots) \\ &\leq U(k_0, k_1^*) + \beta V(k_1^*) \\ &\leq \max_{k \in \Gamma(k_0)} \{U(k_0, k) + \beta V(k)\} \end{aligned}$$

In the other hand, let $\bar{k}_1 \in \Gamma(k_0)$, there exist $(\bar{k}_2, \bar{k}_3, \ldots) \in \Pi(\bar{k}_1)$ such that

$$V(\bar{k}_1) = \max_{k \in \Pi(\bar{k}_1)} \sum_{t=0}^{+\infty} \beta^t U(k_t, k_{t+1}) = \sum_{t=1}^{+\infty} \beta^{t-1} U(\bar{k}_t, \bar{k}_{t+1})$$

We have:

$$\begin{aligned} U(k_0, \bar{k_1}) + \beta V(\bar{k_1}) &= U(k_0, \bar{k_1}) + \beta \sum_{t=1}^{+\infty} \beta^{t-1} U(\bar{k}_t, \bar{k}_{t+1}) \\ &= \sum_{t=0}^{+\infty} \beta^{t-1} U(\bar{k}_t, \bar{k}_{t+1}) \\ &= v(k_0, \bar{k_1}\bar{k_2}, \dots) \\ &\leq V(k_0) \\ \Rightarrow \max_{k \in \Gamma(k_0)} \{U(k_0, k) + \beta V(k)\} &\leq V(k_0) \end{aligned}$$

Hence

$$V(k_0) = \max_{k' \in \Gamma(k_0)} \{U(k_0, k') + \beta V(k')\}$$

□

Remark 4.1 (Growth model with elastic labor - Continued)**.** *Let us consider the problem with elastic labor* (Q)*. For* $k_0 \geq 0$*, define by* $V^Q : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *the value function, i.e:*

$$V(k_0) = \max_{\mathbf{k} \in \Pi^Q(k_0)} \sum_{t=0}^{\infty} \beta^t U(k_t, k_{t+1})$$

The value function V^Q *has the same properties with the value function* V *of problem (P):*

- V^Q *is strictly increasing, strictly concave.*
- V^Q *is upper semicontinuous. By adding the consumption that* $u(0, l) = 0$ *then* V^Q *is continuous.*
- *The value function* V^Q *solves the Bellman equation, i.e*

$$\forall k_0 \geq 0, \quad V^Q(k_0) = \max\{U^Q(k_0, k_1) + \beta V^Q(k_1) : k_1 \in \Gamma^Q(k_0)\}$$

5 Properties of Optimal Paths

To establish additional properties of the optimal solution, we need to impose following assumption.

Assumption P7. $F_{kl}(k, l) \geq 0$ *forall* $k > 0, \ 0 < l < L$.

The results in the following lemma are crucial in establishing important properties of optimal paths.

Lemma 5.1. *Under assumptions* **P1-P7***, U is twice differentiable and*

$$\frac{\partial^2 U(k,y)}{\partial y \partial k} > 0$$

Proof. See appendix. □

Proposition 5.1. *Under assumptions* **P1-P7***, the optimal capital sequence $(k_t^*)_t$ from k_0 is monotonic.*

Proof. [2]

Consider the Value Function

$$V(k_0) = \max_{k_{t+1}\in\Gamma(k_t)} \sum_{t=0}^{\infty} \beta^t U(k_t, k_{t+1}), \quad k_0 \text{ is given}$$

Suppose $k_0 < k_0'$ and let (k_t) and (k_t') be optimal paths for k_0, k_0' respectively.

The Bellman equation gives:

$$\begin{array}{rcccl} V(k_0) & = & U(k_0,k_1) + \beta V(k_1) & \geq & U(k_0,k_1') + \beta V(k_1') \\ V(k_0') & = & U(k_0',k_1') + \beta V(k_1') & \geq & U(k_0',k_1) + \beta V(k_1) \end{array}$$

This implies

$$U(k_0,k_1) + U(k_0',k_1') \geq U(k_0,k_1') + U(k_0',k_1) \tag{11}$$

Since the indirect utility function is twice continuously differentiable we can rewrite as following:

$$U(k_0,k_1') = \int_{k_1}^{k_1'} \frac{\partial U(k_0,y)}{\partial y} dy + U(k_0,k_1)$$

$$U(k_0',k_1) = \int_{k_1'}^{k_1} \frac{\partial U(k_0',y)}{\partial y} dy + U(k_0',k_1')$$

Substituting to (11) we then have

$$\int_{k_1}^{k_1'} \frac{\partial U(k_0,y)}{\partial y} dy + \int_{k_1'}^{k_1} \frac{\partial U(k_0',y)}{\partial y} dy \leq 0$$

implies

$$0 \leq \int_{k_1}^{k_1'} \left[\frac{\partial U(k_0',y)}{\partial y} - \frac{\partial U(k_0,y)}{\partial y} \right] dy = \int_{k_1}^{k_1'} \left[\int_{k_0}^{k_0'} \frac{\partial^2 U(x,y)}{\partial x \partial y} dx \right] dy$$

[2] We adapt the proof given in [8] of Benhabib and Nishimura

Hence

$$\int_{k_1}^{k_1'}\int_{k_0}^{k_0'} \frac{\partial^2 U(x,y)}{\partial x \partial y} dxdy \geq 0$$

From lemma 5.1 we have $\dfrac{\partial^2 U(x,y)}{\partial x \partial y} > 0$, this implies $k_1 \leq k_1'$. Setting $k_0 = k_{t-1}, k_0' = k_t$ for all $t \geq 1$, we then receive that the optimal path (k_t) is monotonic.

□

Proposition 5.2. *Assume* **P1-P7***. If* $\mathbf{k}^* = (k_t^*)_{t=0}^{\infty}$ *is optimal and satisfies* $(k_t^*, k_{t+1}^*) \in int(Graph\Gamma)$, $\forall t$, *then* $\mathbf{k}^*$ *satisfies the Euler equation:*

$$U_2(k_t^*, k_{t+1}^*) + \beta U_1(k_{t+1}^*, k_{t+2}^*) = 0,\ \forall t \geq 0$$

where U_1, U_2 *denote the derivatives of* U *with respect to the first and the second variables.*

Proof. Let $\mathbf{k}^*$ is optimal. Denote:

$$E(k_{t+1}) = U(k_t^*, k_{t+1}) + \beta U(k_{t+1}, k_{t+2}^*)$$

We have E is differentiable and

$$E'(k_{t+1}) = U_2(k_t^*, k_{t+1}) + \beta U_1(k_{t+1}, k_{t+2}^*)$$

Since $(k_t^*, k_{t+1}^*) \in int(\text{Graph}\Gamma)$, $(k_{t+1}^*, k_{t+2}^*) \in int(\text{Graph}\Gamma)$, there exists a neighborhood of k_{t+1}^* such that $(k_t^*, y) \in int(\text{Graph}\Gamma)$, $(y, k_{t+2}^*) \in int(\text{Graph}\Gamma)$ for all y in this neighborhood. We define the sequence

$$z = (k_0, k_1^*, \ldots, k_t^*, y, k_{t+1}^*, \ldots) \in \Pi(k_0)$$

Because of optimality of k^* we have $v(k^*) \geq v(z)$, this implies:

$$\sum_{t=0}^{+\infty} \beta^t U(k_t^*, k_{t+1}^*) \geq \sum_{t=0}^{+\infty} \beta^t U(z_t, z_{t+1})$$
$$\Rightarrow \beta^t U(k_t^*, k_{t+1}^*) + \beta^{t+1} U(k_{t+1}^*, k_{t+2}^*) \geq \beta^t U(y_t^*, y) + \beta^{t+1} U(y, k_{t+2}^*)$$
$$\Rightarrow U(k_t^*, k_{t+1}^*) + \beta U(k_{t+1}^*, k_{t+2}^*) \geq U(y_t^*, y) + \beta U(y, k_{t+2}^*)$$
$$\Rightarrow E(k_{t+1}^*) \geq E(y) \quad \text{for every } y \text{ in a neighborhood of } k_{t+1}^*$$

It means that k_{t+1}^* is a maximum in this neighborhood. Hence $E'(k_{t+1}^*) = 0$. The result then follows. □

Proposition 5.3. *Assume* **P1-P7***. If* $\mathbf{k}^* = (k_t^*)_{t=0}^{\infty} \in \Pi(k_0)$ *which satisfies all of the following conditions:*

(i) $(k_t^*, k_{t+1}^*) \in int(Graph(\Gamma))$, $\forall t \geq 0$

(ii) Euler equation: $U_2(k_t^*, k_{t+1}^*) + \beta U_1(k_{t+1}^*, k_{t+2}^*) = 0,\ \forall t \geq 0$

(iii) Transversality condition: $\lim_{t\to\infty} \beta^t U_1(k_t^*, k_{t+1}^*) \cdot k_t^* = 0$

then $\mathbf{k}^*$ *is optimal.*

Proof. Let $\mathbf{k}^* \in \Pi(k_0)$ satisfies the conditions (i)-(iii). Let $k \in \Pi_(k_0)$. We will prove that

$$\Delta := \sum_{t=0}^{\infty} \beta^t U(k_t^*, k_{t+1}^*) - \sum_{t=0}^{\infty} \beta^t U(k_t, k_{t+1}) \geq 0$$

Indeed, by the concavity and differentability of U we have:

$$\begin{aligned}
\Delta_T &:= \sum_{t=0}^{T} \beta^t U(k_t^*, k_{t+1}^*) - \sum_{t=0}^{T} \beta^t U(k_t, k_{t+1}) \\
&\geq \sum_{t=0}^{T} \beta^t \Big[U_1(k_t^*, k_{t+1}^*) \cdot (k_t^* - k_t) + U_2(k_t^*, k_{t+1}^*) \cdot (k_{t+1}^* - k_{t+1}) \Big] \\
&= \sum_{t=0}^{T-1} \beta^t \Big[U_2(k_t^*, k_{t+1}^*) + \beta U_1(k_{t+1}^*, k_{t+2}^*) \Big] (k_{t+1}^* - k_{t+1}) + \beta^T U_2(k_T^*, k_{T+1}^*) \cdot (k_{T+1}^* - k_{T+1})
\end{aligned}$$

since the Euler equation holds, then:

$$\begin{aligned}
\Delta_T &\geq \beta^T U_2(k_T^*, k_{T+1}^*) \cdot (k_{T+1}^* - k_{T+1}) \geq \beta^T U_2(k_T^*, k_{T+1}^*) \cdot k_{T+1}^* \\
&= -\beta^{T+1} U_1(k_{T+1}^*, k_{T+2}^*) \cdot k_{T+1}^*
\end{aligned}$$

hence

$$\Delta = \lim_{T\to\infty} \Delta_T \geq 0$$

implies that $\mathbf{k}^*$ is optimal. □

Corollary 5.1. *Assume* **P1-P7**. *Let* $\mathbf{k}^*$ *is optimal path of capital, then the optimal path of labour is the sequence* $\mathbf{l}^* = (l_t^*)_{t\geq 0}$ *where* l_t^* *is unique solution of the equation:*

$$k_{t+1}^* = F(k_t^*, l_t^*) + (1-\delta)k_t^* + \frac{l_t^* - L}{a}$$

The final step in this section is to establish convergence of the optimal solution $(\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*)$ to a unique non trivial optimal steady state (c^s, k^s, l^s).

We know that the optimal sequence $\mathbf{k}^*$ is monotonic and bounded, so it converges to some k^s. Associated with this k^s, there exists (c^s, l^s) that solves the maximization problem defined in definition 3.1. Taking the limits in Euler equation, we get

$$U_2(k^s, k^s) + \beta U_1(k^s, k^s) = 0.$$

Substituting the expressions for U_1 and U_2 from proposition 3.1 we obtain

$$F_1(k^s, l^s) + (1-\delta) = \frac{1}{\beta}.$$

Lemma 5.2. *Assume* **P1-P7**. *Let* (k^s, l^s) *satisfies*

$$F_1(k^s, l^s) + (1 - \delta) = \frac{1}{\beta}$$

Define the sequence $\mathbf{k}^s$ *by:*

$$\mathbf{k}^s = (k_t^s)_{t \geq 1} : \ k_t^s = k^s \ \forall t \geq 1$$

Then $k^s \in \Gamma(k^s)$ *and the sequence* $\mathbf{k}^s$ *satisfies Euler equation and transversality condition for all* $t \geq 1$.

Proof. Since $F_1(k^s, l^s) = \frac{1}{\beta} - 1 + \delta < F_1(0, L)$ then $k^s > 0, l^s < L$. This implies $k^s > (1-\delta)k^s - \frac{L}{a}$. Since $F_1(k^s, l^s) - \delta = \frac{1}{\beta} - 1 > 0$, it is easy to see that $k^s < F(k^s, l^s) + (1-\delta)k^s < F(k^s, L) + (1 - \delta)k^s$. Hence $k^s \in \Gamma(k^s)$. Moreover $k^s \in int(\Gamma(k^s))$.

By the definition of U we have

$$U(k_t, k_{t+1}) = u\left(\frac{L - l_t}{a}\right)$$

where

$$k_{t+1} = F(k_t, l_t) + (1 - \delta)k_t + \frac{l_t - L}{a}$$

we get

$$\frac{dl_t}{dk_t} = -[F_1(k_t, l_t) + (1 - \delta)]\frac{dl_t}{dk_{t+1}}$$

then

$$U_1(k_{t+1}^s, k_{t+2}^s) = -\frac{1}{\beta}U_2(k_t^s, k_{t+1}^s) \ \Rightarrow \ U_2(k_t^s, k_{t+1}^s) + \beta U_1(k_{t+1}^s, k_{t+2}^s) = 0$$

It is obviously see that $\lim_{t\to\infty} \beta^t U_1(k_t^s, k_{t+1}^s) \cdot k_t^s = [U_1(k^s, k^s) \cdot k^s] \lim_{t\to\infty} \beta^t = 0$.

The result then follow. □

Corollary 5.2. *Assume* **P1-P7**. *Let* (c^s, k^s, l^s) *denote the unique nontrivial steady state satisfied*

$$\begin{aligned} c^s + \delta k^s &= F(k^s, l^s) \\ ac^s + l^s &= L \\ F_k(k^s, l^s) &= \delta + \tfrac{1}{\beta} - 1 \end{aligned}$$

Then for all $k_0 > 0$, *the optimal solution* $(\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*)$ *converges to* (c^s, k^s, k^s) *when* t *goes to* $+\infty$.

Remark 5.1 (Growth model with elastic labor - Continued)**.** *We also have the similar results to problem* (Q).

- *Let* $k_0 \geq 0$. *The optimal capital sequence* $\mathbf{k}^*$ *from* k_0 *is monotonic.*

- *There exists a unique nontrivial steady state* (c^Q, k^Q, l^Q) *that satisfies*

$$F_k(k^Q, l^Q) + (1-\delta)k^Q = \frac{1}{\beta} \quad \text{and} \quad c^Q = F(k^Q, l^Q) - \delta k^Q$$

- *For all* $k_0 > 0$*, the optimal solution* $(\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*)$ *has the property that converges to* (c^Q, k^Q, l^Q) *when* t *converges to* $+\infty$*.*

Example. Consider an economy with period utility function of the following form: $u(c) = c^\alpha,\ \alpha \in (0,1)$. Technology is Cobb-Douglas, $F(k,l) = k^\gamma l^{1-\gamma}$, where $\gamma \in (0,1)$. For simplicity assume that $\delta = 1$ (full depreciation) and $L = 1$.

It is easy to check that the steady state of this economy is:

$$c^P = \frac{\tau^\gamma - \tau}{a(\tau^\gamma - \tau) + 1}$$
$$k^P = \frac{\tau}{a(\tau^\gamma - \tau) + 1}$$
$$l^P = \frac{1}{a(\tau^\gamma - \tau) + 1}$$

where $\tau = (\beta\gamma)^{\frac{1}{1-\gamma}}$.

By comparison with problem (Q), consider the period utility function $u^Q(c, \mathfrak{L}) = c^\alpha + \mathfrak{L}^\alpha,\ \alpha \in (0,1)$. The steady state is:

$$c^Q = \frac{\eta(\tau^\gamma - \tau)}{\tau^\gamma - \tau + \eta}$$
$$k^Q = \frac{\tau\eta}{\tau^\gamma - \tau + \eta}$$
$$l^Q = \frac{\eta}{\tau^\gamma - \tau + \eta}$$
$$\mathfrak{L}^Q = \frac{\tau^\gamma - \tau}{\tau^\gamma - \tau + \eta}$$

where $\tau = (\beta\gamma)^{\frac{1}{1-\gamma}},\ \eta = [(1-\gamma)\tau^\gamma]^{\frac{1}{1-\alpha}}$.

Appendix

Proof of the Lemma 3.2

We claim that there exist $t \geq 0$ such that $l_t^* > 0$. Indeed, if $l_t^* = 0$ for all $t \geq 0$ then the constraint (1) becomes:

$$\frac{L}{a} + k_{t+1}^* - (1-\delta)k_t^* \leq 0 \quad \Leftrightarrow \quad k_{t+1}^* \leq (1-\delta)k_t^* - \frac{L}{a} \quad \Rightarrow \quad k_{t+1}^* \leq k_t^*$$

The sequence $(k_t^*)_t$ is decreasing and bounded below by 0, so k_t^* converges to $k^* \geq 0$ satisfied:

$$k^* \leq (1-\delta)k^* - \frac{L}{a} \quad \Rightarrow \quad k^* \leq -\frac{L}{a\delta} < 0$$

This contradiction shows that there exists $l_t^* > 0$.

Assume $(c_t^*, k_{t+1}^*, l_t^*)$ is an optimal solution of (P) and

$$k_{t+1}^* < F(k_t^*, l_t^*) + (1-\delta)k_t^* + \frac{l_t^* - L}{a}$$

If $l_t^* > 0$, we can reduce l_t^* a small amount $\varepsilon > 0$ such that $l_t^* - \varepsilon \geq 0$ and

$$k_{t+1}^* \leq F(k_t^*, l_t^* - \varepsilon) + (1-\delta)k_t^* + \frac{l_t^* - \varepsilon - L}{a}$$

Then $((c_t^* + \frac{\varepsilon}{a}), (l_t^* - \varepsilon), (k_{t+1}^*))$ satisfies all constraints and

$$\sum \beta^t u(\frac{L - (l_t^* - \varepsilon)}{a}) > \sum \beta^t u(\frac{L - l_t^*}{a})$$

contradiction with the optimal solution of (l_t^*). So the equation (9) is satisfied with any t such that $l_t^* > 0$.

We will now prove that (9) holds for every $t \geq 0$. In contrary, suppose that there exists t_0 such that (9) does not bind. It implies $l_{t_0}^* = 0$. Without loss of generality, we can assume that $l_0^* = 0,\ l_1^* > 0$. We have:

$$k_1^* < F(k_0^*, l_0^*) + (1-\delta)k_0^* + \frac{l_0^* - L}{a} \tag{12}$$

$$k_2^* = F(k_1^*, l_1^*) + (1-\delta)k_1^* + \frac{l_1^* - L}{a} \tag{13}$$

We can increase k_1 from (12) such that:

$$(k_1^* + \varepsilon_k) \leq F(k_0^*, l_0^*) + (1-\delta)k_0^* + \frac{l_0^* - L}{a}, \quad \varepsilon_k > 0$$

From (13) we have:

$$k_2^* < F(k_1^* + \varepsilon_k, l_1^*) + (1-\delta)(k_1^* + \varepsilon_k) + \frac{l_1^* - L}{a}$$

Since $l_1^* > 0$, we can decrease l_1^* such that:

$$k_2^* \leq F(k_1^* + \varepsilon_k, (l_1^* - \varepsilon_l)) + (1-\delta)(k_1^* + \varepsilon_k) + \frac{(l_1^* - \varepsilon_l) - L}{a}, \quad \varepsilon_l > 0, l_1^* - \varepsilon_l > 0$$

Consider $(\bar{k}_t, \bar{l}_t)$:

$$\bar{k}_t = \begin{cases} k_1^* + \varepsilon_k, & t = 1 \\ k_t^*, & t \neq 1 \end{cases} \qquad \bar{l}_t = \begin{cases} l_1^* - \varepsilon_l, & t = 1 \\ l_t^*, & t \neq 1 \end{cases}$$

then $(\bar{k}_t, \bar{l}_t)$ satisfies all constrains of problem (P') and

$$\sum_{t=0}^{\infty} \beta^t u\left(\frac{L-\bar{l}_t}{a}\right) > \sum_{t=0}^{\infty} \beta^t u\left(\frac{L-l_t}{a}\right)$$

a contradiction with optimal of (k_t^*, l_t^*).

Hence at optimum

$$k_{t+1}^* = F(k_t^*, l_t^*) + (1-\delta)k_t^* + \frac{l_t^* - L}{a}$$

we obtain:

$$k_{t+1}^* \leq F(k_t^*, L) + (1-\delta)k_t^* + \frac{L-L}{a} = F(k_t^*, L) + (1-\delta)k_t^*$$
$$k_{t+1}^* \geq F(k_t^*, 0) + (1-\delta)k_t^* + \frac{0-L}{a} = (1-\delta)k_t^* - \frac{L}{a}$$

□

Proof of the Lemma 3.3

i) We consider equation (variable k):

$$F(k, L) + (1-\delta)k = k \quad \Leftrightarrow \quad F(k, L) = \delta k \tag{14}$$

Because of concavity of F and the assumption $F_k(\infty, L) = 0$, we have two cases:

• Case 1. The equation (14) has unique solution $\bar{k} = 0$. Then we have

$$F(k, L) + (1-\delta)k \leq k \quad \Rightarrow \quad k' \leq k$$

• Case 2. The equation (14) has two solutions $\{0, \bar{k} > 0\}$. If $k \leq \bar{k}$ then $k' \leq \bar{k}$. If $k > \bar{k}$ then $k' < k$. We always have $k' \leq \max(\bar{k}, k)$

ii) We have for all $k \geq 0$: $(1-\delta)k \in \Gamma(k)$, so $\Gamma(k) \neq \emptyset$. It is easy to see that $\Gamma(k)$ is compact convex. Moreover

$$\Gamma(0) = \{k' \geq 0 : -\frac{L}{a} \leq k' \leq 0\} = \{0\}$$

Fix $k \geq 0$. For any $k' \in \Gamma(k)$ and for any sequence $\{k_n\} \subset \mathbb{R}_+$ converging to k, we consider sequence $\{k_n'\}$:

$$k_n' = \frac{F(k_n, L) + \frac{L}{a}}{F(k, L) + \frac{L}{a}}\left[k' - (1-\delta)k - \frac{L}{a}\right] + (1-\delta)k_n - \frac{L}{a}$$

One can easily check that $k'_n \to k'$ and

$$(1-\delta)k_n - \frac{L}{a} \leq k'_n \leq F(k_n, L) + (1-\delta)k_n \quad \Rightarrow k'_n \in \Gamma(k_n)$$

It shows that the correspondence Γ is lower semi-continuous.

In the other hand, for any sequence $\{k_n\} \subset \mathbb{R}_+$ converging to k, for any sequence $\{k'_n\}$ with $k'_n \in \Gamma(k_n), \forall n$, we have

$$(1-\delta)k_n - \frac{L}{a} \leq k'_n \leq F(k_n, L) + (1-\delta)k_n$$

Because of assumption P6, there exist $\bar{k} \geq 0$ such that $k'_n \leq \max(\bar{k}, k_n)$. The sequence $\{k_n\}$ is converged so it is bounded, then the sequence $\{k'_n\}$ is bounded. Hence it has a subsequence $\{k'_{n_m}\}$ converging to k'. It is easy to see that $k' \in \Gamma(k)$. Then the correspondence Γ is upper semi-continuous.

The result then follows.

iii) From the first part of this lemma we have, for $k = (k_0, k_1, \ldots) \in \Pi(k_0)$:

$$\begin{aligned} &k_1 \leq \max(\bar{k}, k_0) \\ &k_2 \leq \max(\bar{k}, k_1) \leq \max(\bar{k}, k_0) \\ &\ldots \end{aligned}$$

Denote $K = \max(\bar{k}, k_0)$ we have $k_t \leq K, \forall t \geq 0$. Hence the set $\Pi(k_0)$ is bounded.

Because of continuity of Γ we can easily check that $\Pi(k_0)$ is compact.

□

Proof of the Proposition 3.1

1) The Kuhn-Tucker first-order conditions are:

$$u'_c(c^*) - \lambda - a\mu + \xi_1 = 0 \tag{15}$$

$$\lambda F'_l(k, l^*) - \mu + \xi_2 = 0 \tag{16}$$

$$\lambda \geq 0, \ \lambda[F(k, l^*) + (1-\delta)k - c^* - y] = 0 \tag{17}$$

$$\mu \geq 0, \ \mu[L - ac^* - l] = 0 \tag{18}$$

$$\xi_1 \geq 0, \ \xi_1 c^* = 0 \tag{19}$$

$$\xi_2 \geq 0, \ \xi_2 l^* = 0 \tag{20}$$

The Inada condition of utility function implies that $c^* > 0$, and then $\xi_1 = 0$. The Inada condition on labor's marginal productivity implies that $l^* > 0$, $\xi_2 = 0$. The strict increasingness of u and F imply that $\lambda > 0$ and $\mu > 0$. From (18) we then have $l^* \in (0, L)$. The uniqueness of solution follows from the strict concavity of u.

2) Consider

$$\begin{aligned}\mathfrak{L} = u(c) \quad &+ \quad \lambda[F(k,l) + (1-\delta)k - c - y] \\ &+ \quad \mu[L - ac - l]\end{aligned}$$

We have

$$u'_c(c^*) - \lambda - a\mu = 0 \tag{21}$$
$$\lambda F'_l(k, l^*) - \mu = 0 \tag{22}$$
$$c^* + y = F(k, l^*) + (1-\delta)k \tag{23}$$
$$ac^* + l^* = L \tag{24}$$

Differentiating the above equations with respect to k and y gives:

$$\begin{aligned}&u''_{cc}dc^* - d\lambda - ad\mu = 0 \\ &F'_l d\lambda + \lambda[F''_{lk}dk + F''_{ll}dl^*] - d\mu = 0 \\ &dc^* + dy - [F'_k dk + F'_l dl^*] - (1-\delta)dk = 0 \\ &adc^* + dl^* = 0\end{aligned}$$

Writting these equations in a matrix form we get:

$$\underbrace{\begin{pmatrix} u''_{cc} & 0 & -1 & -a \\ 0 & \lambda F''_{ll} & F'_l & -1 \\ 1 & -F'_l & 0 & 0 \\ a & 1 & 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} dc^* \\ dl^* \\ d\lambda \\ d\mu \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\lambda F''_{lk} & 0 \\ F'_k + 1 - \delta & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} dk \\ dy \end{pmatrix}$$

The determinant of matrix A:

$$det(A) = (1 + aF'_l)^2$$

Since the production function F is strictly increasing then $det(A) > 0$, hence A is invertible. It implies that $c^*(k,y), l^*(k,y), \lambda(k,y), \mu(k,y)$ are continuously differentiable in a neighborhood of (k,y).

By the Envelope Theorem we have:

$$\begin{aligned}\frac{\partial U(k,y)}{\partial k} &= \frac{\partial \mathfrak{L}}{\partial k}(c^*, l^*, \lambda, \mu) = \lambda[F'_k(k, l^*) + 1 - \delta] \\ \frac{\partial U(k,y)}{\partial y} &= \frac{\partial \mathfrak{L}}{\partial y}(c^*, l^*, \lambda, \mu) = -\lambda\end{aligned}$$

From (21) and (22) we have

$$\lambda = \frac{u'_c(c^*)}{1 + aF'_l(k, l^*)}; \quad \mu = \frac{u'_c(c^*)F'_l(k, l^*)}{1 + aF'_l(k, l^*)}$$

3) The results are directly followed by the last part.

4) It is easy to see that GraphΓ is a convex set.

Given $(k_1, k_1') \in \mathbb{R}^2_+, (k_2, k_2') \in \mathbb{R}^2_+$, we have

$$U(k_1, k_1') = u(\frac{L - l_1}{a}),\ k_1' = F(k_1, l_1) + (1-\delta)k_1 + \frac{l_1 - L}{a}$$
$$U(k_2, k_2') = u(\frac{L - l_2}{a}),\ k_2' = F(k_2, l_2) + (1-\delta)k_2 + \frac{l_2 - L}{a}$$

For $\lambda \in [0,1]$, define $k_\lambda = \lambda k_1 + (1-\lambda)k_2$; $k_\lambda' = \lambda k_1' + (1-\lambda)k_2'$. We have $(k_\lambda, k_\lambda') \in$ GraphΓ and

$$U(k_\lambda, k_\lambda') = u(\frac{L - l_\lambda}{a}),\ k_\lambda' = F(k_\lambda, l_\lambda) + (1-\delta)k_\lambda + \frac{l_\lambda - L}{a}$$

this implies

$$\lambda k_1' + (1-\lambda)k_2' = F(\lambda k_1 + (1-\lambda)k_2, l_\lambda) + (1-\delta)(\lambda k_1 + (1-\lambda)k_2) + \frac{l_\lambda - L}{a}$$

then

$$F\Big(\lambda k_1 + (1-\lambda)k_2,\ \lambda l_1 + (1-\lambda)l_2\Big) + \frac{\Big(\lambda l_1 + (1-\lambda)l_2\Big) - L}{a} \geq F\Big(\lambda k_1 + (1-\lambda)k_2, l_\lambda\Big) + \frac{l_\lambda - L}{a}$$
$$\Rightarrow l_\lambda \leq \lambda l_1 + (1-\lambda)l_2$$

Using the strictly concavity of u we then have:

$$\begin{aligned}
U\Big(\lambda k_1 + (1-\lambda)k_2,\ \lambda k_1' + (1-\lambda)k_2'\Big) &= u\Big(\frac{L - l_\lambda}{a}\Big) \\
&\geq u\Big(\frac{L - (\lambda l_1 + (1-\lambda)l_2)}{a}\Big) \\
&= u\Big(\lambda\frac{L - l_1}{a} + (1-\lambda)\frac{L - l_2}{a}\Big) \\
&> \lambda u\Big(\frac{L - l_1}{a}\Big) + (1-\lambda)u\Big(\frac{L - l_2}{a}\Big) \\
&= \lambda U(k_1, k_1') + (1-\lambda)U(k_2, k_2')
\end{aligned}$$

In other words, we have proved that U is strictly concave.

Moreover, because U is non-negative then there exist $A \geq 0, B \geq 0$ such that $U(k, k') \leq A + B(k + k')$

□

Proof of the Lemma 5.1

Using the argument applied in proof of the proposition 3.1, by tedious computations:

$$\frac{\partial \lambda}{\partial k} = \frac{1}{(1 + aF_l')^2}\left[-a\lambda(1 + aF_l')F_{lk}'' + (a^2\lambda F_{ll}'' + u_{cc}'')(F_k' + 1 - \delta)\right]$$

Consider the expression in parentheses. Observe that the concavity of F and strict concavity of u imply $(a^2\lambda F''_{ll} + u''_{cc}) < 0$. Then the second term is strictly negative. By the assumption P7 we receive that the first term is strictly negative. Hence

$$\frac{\partial \lambda}{\partial k} < 0$$

It follows that

$$\frac{\partial^2 U(k,y)}{\partial y \partial k} = -\frac{\partial \lambda}{\partial k} > 0$$

□

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