General existence of competitive equilibrium in the growth model with an endogenous labor-leisure choice and unbounded capital stock*

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Abstract

We prove the existence of competitive equilibrium in the neoclassical growth model with elastic labor supply under general conditions which allow for unbounded capital stock (including case of linear technology) and corner solutions. We do not impose Inada conditions, strict concavity, homogeneity, differentiability, or require a maximum sustainable capital stock. We give examples to illustrate the violation of the conditions used in earlier existence results but where a competitive equilibrium can be shown to exist following the approach in this paper.

Keywords: Optimal growth, Competitive equilibrium, Lagrange multipliers, Elastic labor supply, Inada conditions, Linear technology, Unbounded capital stock.
JEL Classification: *C61, D51, E13, O41*

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1 Introduction

The optimal growth model is one of the main frameworks in macroeconomics. While variations of the model with inelastic labor supply are used widely in growth theory, the version with elastic labor supply is the canonical model in business cycle models, both for exogenous and endogenous fluctuations. Despite the central place of the model with elastic labor supply in dynamic general equilibrium models, existence of competitive equilibrium in general settings has proved to be a challenge. Recent results of existence of equilibrium for this model use strong conditions for existence (Coleman (1997), Datta, et al. (2002), Greenwood and Huffman(1995), Le Van and Vailakis (2004), and Yano (1989, 1990, 1998)). This paper establishes existence of equilibrium under very weak conditions: neither Inada conditions, nor strict concavity, nor differentiability, nor constant returns to scale (or more generally, homogeneity), nor restrictions on cross-partials of the utility functions, nor interiority assumptions. In particular, we prove existence of equilibrium (Example 3) where the consumption of the produced good is zero and capital stock is **unbounded**.

The approach taken in this paper is a direct method based on existence of Lagrange multipliers to the optimal problem and their representation as a summable sequence. The problem with inelastic labor supply was considered by Le Van and Saglam (2004). This approach uses a separation argument where the multipliers are represented in the dual space $(\ell^\infty)'$ of the space of bounded sequences ℓ^∞. While one would like the multipliers and prices to lie in ℓ^1, it is not the dual space. In the previous work on competitive equilibrium following Peleg and Yaari (1970), the representation theorems followed separation arguments applied to arbitrary vector spaces (see Bewley (1972), Aliprantis, et al. (1997), Dana and Le Van (1991)). The Le Van and Saglam (2004) approach also uses a separation argument but imposes restrictions on the asymptotic behavior of the objective functional and constraint functions which are easily shown to be satisfied in standard models. This is related to Dechert (1982). There is a difficulty in going from the inelastic labor supply (e.g. Le Van and Saglam (2004)) to the elastic labor supply model: While one may be able to show that the optimal capital stock is strictly positive, one cannot be sure that the optimal labor supply sequence is strictly positive. Thus, the paper by Le Van and Vailakis (2004) which took the approach of decentralizing the optimal solution via prices as marginal utilities had to make additional strong conditions on the utility function (which fails in homogeneous utility functions such as those of the Cobb-Douglas class) to ensure that the labor supply sequence remains strictly positive. We show the Lagrange multipliers to the social planners problem are a summable sequence. Thus, we can directly use these to decentralize the optimal solution and not have to make strong assumptions to ensure interiority of the optimal plan.[1] Thus, the Inada conditions do not have to be assumed. As the separation theorem does not require strict concavity or differentiability, these strong assumptions on utility functions can be dropped. This is especially important as an important specification of preferences in applied macroeconomics models are quasi-linear utility with linear utility of leisure where strict concavity and Inada conditions are violated. The linear specification also results in the planners problem in models with indivisible labor (see Hansen (1985), and Rogerson (1988)). Note that while calibrated models essentially result in an interior solution, for the general specification of these models we cannot rule out the case where either labor or leisure is equal to zero in some or all the time periods. In fact, the problem is more fundamental. While for some examples we can calculate the equilibrium allocation, we still have to show that there always exist equilibrium prices that are summable so that values have an inner product representation. Being able to calculate a candidate "equilibrium allocation" does not say anything on whether the price system is an equilibrium one. The relaxation of Inada conditions is also important as they may also be violated in utility and production functions of the CES class which are also widely used in the applied literature. Furthermore, there is no need to make any assumption on cross-partial derivatives of the utility function as in Aiyagari, et al. (1992), Coleman (1997), Datta, et al. (2002), Greenwood and Huffman (1995), and Le Van, et al. (2007).[2] Thus, whether labor supply is backward bending or not, and whether consumption is inferior or not plays no role in existence of equilibrium. As only convexity and not differentiability is required for the separation theorem we are also able to cover Leontieff and more generally linear activity analysis models that are not covered by the existing results.

Yano (1984, 1990, 1998) also studies existence of competitive equilibrium with endogenous labor under general conditions. There are both produced input/consumption goods (i.e. capital) and non-produced input/consumption goods (which can be interpreted as labor/leisure). While the conditions in these papers weaken the conditions used in Bewley (1982) they do not cover our existence result. Yano (1984) has the most general specification and is the closest to our assumptions. It does not use differentiability (and hence, Inada conditions). It also does not use interiority assumptions in Bewley (1982). However, it makes assumptions A.14-A.17 that we do not have to make. In our results as we are concerned only with the existence issue we allow for corner solutions.

[1] Goenka, et al. (2012) in a model with heterogeneous agents also assume Inada conditions. While there is an interior solution for aggregate variables, the consumption and leisure of the more impatient consumers converge to zero as time tends to infinity.

[2] These papers essentially show the isomorphism of the dynamic problem with endogenous leisure to one without endogenous leisure, and the assumptions are used to show monotonicity of the optimal capital path which combined with the static labor-leisure choice gives existence in the original problem.

This is ruled out by A.14-A.17 in Yano (1984). Yano (1990) assumes continuous differentiability of the production function (A.1), utility function (A.5), and Inada conditions on the utility function (A.7). There is also an interiority condition (g, p.37) that says that all countries (firms) produce a positive output in equilibrium. Our paper does not use these conditions (see U1-U2, F1-F2, p.5.) In fact, in Example 2 we show under these conditions it is possible in a competitive equilibrium while there is positive output it is entirely consumed. Yano (1998) also assumes continuous differentiability and Inada conditions for utility (Assumption 1) and production functions (Assumption 2), which are not assumed in our paper.

The organization of the paper is as follows. Section 2 describes the model and provides the sufficient conditions on the objective function and the constraint functions so that Lagrangean multipliers can be represented by an ℓ^1_+ sequence of multipliers in optimal growth model with leisure in the utility function. In section 3, we prove the existence of competitive equilibrium in a model with a representative agent by using these multipliers as sequences of prices and wages. Section 4 gives examples with corner solutions to illustrate that a competitive equilibrium will still exist using the main result of the paper. Section 5 contains a further discussion of the literature and concludes.

2 Lagrange multipliers in the optimal growth model

We study the optimal growth model with an endogenous labor-leisure choice. Thus, it is an economy where the representative consumer has preferences defined over processes of consumption and leisure described by the discounted utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t).$$

In each period $t \geq 0$, the consumer faces two resource constraints given by

$$\begin{aligned} c_t + k_{t+1} &\leq F(k_t, L_t) + (1-\delta)k_t, \\ l_t + L_t &= 1, \end{aligned}$$

where F is the production function, u the period utility function, k_t the capital stock and c_t consumption in period t, $\delta \in (0,1)$ is the depreciation rate of capital stock, β the discount factor, l_t is leisure and L_t is labor. These constraints restrict allocations of commodities and time for the leisure.

Formally, the problem of the representative consumer is stated as follows:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$\begin{aligned} \text{s.t. } c_t + k_{t+1} &\leq F(k_t, 1-l_t) + (1-\delta)k_t, \ \forall t \geq 0 \\ c_t &\geq 0, \ k_t \geq 0, \ l_t \geq 0, \ 1-l_t \geq 0, \ \ \forall t \geq 0 \\ k_0 &\geq 0 \text{ is given.} \end{aligned}$$

We make a set of assumptions on preferences and the production technology. The assumptions on the period utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ are:

Assumption U1: *u is continuous, concave, increasing on $\mathbb{R}^2_+$, and strictly increasing on $\mathbb{R}^2_{++}$.*

It is worth discussing this assumption. In the optimal growth model where there is no labor-leisure choice, the conventional assumption on the one-period utility function is u is strictly increasing in R_+ (Aliprantis, et al (1998), p. 673, and Le Van and Saglam (2004), Assumption 3, p. 400). This does not extend to our model and thus, we assume u to be *increasing* in R^2_+.[3]

Assumption U2: $u(0,0) = 0$.

The assumptions on the production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ are as follows:

Assumption F1: $F(0,0) = 0, F$ *is continuous, concave, increasing on $\mathbb{R}^2_+$, and strictly increasing on $\mathbb{R}^2_{++}$.*

Assumption F2: There exist a positive number M such that $F(k_t, 1) \leq M k_t$. Moreover, $F_k(0,1) \geq \delta$.

Assumption F3: $(M + 1 - \delta)\beta < 1$.

The assumptions **U1**, **U2**, **F1** are standard. Note we do not assume strict concavity, differentiability or Inada conditions for the utility and production functions. From Assumption **F2**, the capital can be unbounded and allows for a linear technology. The condition $F_k(0,1) \geq \delta$ is standard and is required for the Slater condition to be satisfied. Assumption **F3** ensures the continuity of the objective function (see Becker and Boyd III (1995)).

[3]Let x, y be two vectors of R^n. We write $x \leq y$ if $x_i \leq y_i$ for all i, and $x < y$ if $x_i \leq y_i$ for all i and $x_i < y_i$ for at least one i. A function $u(x)$ is said to be increasing if $u(x) \leq u(y)$ for all $x < y$. It is said to be strictly increasing if $u(x) < u(y)$ for all $x < y$.Consider the standard function of the Cobb-Douglas class: $u(c,l) = \sqrt{cl}$. Let $x(c,l) = (1,0), y(c,l) = (2,0)$. Obviously, $x < y$. However, $u(x) = u(y) = 0$. Thus $u(c,l) = \sqrt{cl}$ is *increasing on R^2_+ and strictly increasing on R^2_{++}* but not *strictly increasing on R^2_+*

We have relaxed some important assumptions in the literature. Bewley (1972) assumes that the production set is a convex cone (Theorem 3). Bewley (1982) assumes the strictly positiveness of derivatives of utility functions on $\mathbb{R}^L_+$ (strictly monotonicity assumption). In our model, the utility functions may not be differentiable in $\mathbb{R}^2_+$.[4] Le Van, et al. (2007) assumed the cross-partial derivative u^i_{cl} has constant sign, $u^i_c(x,x)$ and $u^i_l(x,x)$ are non-increasing in x, production function F is homogenous of degree $\alpha \leq 1$ and $F_{kL} \geq 0$ (Assumptions U4, F4, U5, F5).[5] We have also relaxed the usual assumption $F_k(\infty, 1) < \delta$ used in the existing literature where the capital stock is bounded by a maximum sustainable capital stock (Aliprantis, et al (1998), and Le Van and Saglam (2004)).

A sequence $\{c_t, k_t, l_t\}_{t=0,1,\ldots,\infty}$ is feasible from k_0 if it satisfies the constraints

$$\begin{aligned} c_t + k_{t+1} &\leq F(k_t, 1-l_t) + (1-\delta)k_t, \ \ \forall t \geq 0, \\ c_t &\geq 0, \ k_t \geq 0, \ l_t \geq, \ 1 - l_t \geq 0, \ \ \forall t \geq 0, \\ k_0 &> 0 \text{ is given.} \end{aligned}$$

It is easy to check that, for any initial condition $k_0 > 0$, a sequence $\mathbf{k} = \{k_t\}_{t=0}^{\infty}$ is feasible iff $0 \leq k_{t+1} \leq F(k_t, 1) + (1-\delta)k_t$ for all t. The class of feasible capital paths is denoted by $\Pi(k_0)$. A pair of consumption-leisure sequences $\{\mathbf{c}, \mathbf{l}\} = \{c_t, l_t\}_{t=0}^{\infty}$ is feasible from $k_0 > 0$ if there exists a sequence $\mathbf{k} \in \Pi(k_0)$ that satisfies $0 \leq c_t + k_{t+1} \leq F(k_t, 1-l_t) + (1-\delta)k_t$ and $0 \leq l_t \leq 1$ for all t.

We will extend the l^∞ to include the sequences growing without bound. Denote $H = M + 1 - \delta$. The we define the space l^∞_H which is weighted by H^t as

$$l^\infty_H := \left\{ \mathbf{x} \geq \mathbf{0}, \sup_{t \geq 0} \frac{|x_t|}{H^t} < +\infty \right\}.$$

The dual space of l^∞ is the space ba, the set of finitely additive set functions on $\mathbb{N}$. The dual space of l^∞_H is the weighted ba space, denoted by ba_H. Le Van and Saglam (2004) provided a result where the Lagrange multipliers of social planner problem lie in a subspace of ba, which is identified with the space l^1. Since the weighted spaces l^∞_H and ba_H are isometrically isomorphic to spaces l^∞ and ba, the mathematical techniques used in Le Van and Saglam (2004) can be applied in our model.

For any $\mathbf{k} \in \Pi(k_0)$, we have $0 \leq k_{t+1} \leq F(k_t, 1) + (1-\delta)k_t \leq (M + 1 - \delta)k_t \leq H^{t+1}k_0$. Therefore, $\sup \frac{k_t}{H^t} < +\infty$. Thus, $\mathbf{k} \in \ell^\infty_H$ which in turn implies $\mathbf{c} \in \ell^\infty_H$, if $\{\mathbf{c}, \mathbf{k}\}$ is feasible from k_0. From now onwards we will use the denotations $l^\infty_+ = \{\mathbf{x} \in l^\infty_H \text{ and } \mathbf{x} \geq \mathbf{0}\}$, $l^1_+ =$

[4] Let $F(k, L) = k^\alpha L^{1-\alpha}$, $\alpha \in (0, 1)$. This function is not differentiable even in the extended real numbers at $(0, L)$ or $(k, 0)$ for $L \geq 0, K \geq 0$. The assumptions in Bewley (1982) that $u_c >> 0$, $u_l >> 0$, and D^2u is negative definite on R^2_+ are obviously violated.

[5] See section 5 for a further discussion of assumptions in the literature.

$\{\mathbf{x} \in l^1 \text{ and } \mathbf{x} \geq \mathbf{0}\}$.

Lemma 1. *For any ε, there exists T such that*

$$\sum_{t=T+1}^{\infty} \beta^t u(c_t, l_t) \leq \varepsilon.$$

Proof. We have $0 \leq c_t \leq F(k_t, 1) + (1-\delta)k_t \leq (M+1-\delta)^{t+1}k_0 = H^{t+1}k_0$. Then

$$\sum_{t=T+1}^{\infty} \beta^t u(c_t, l_t) \leq \sum_{t=T+1}^{\infty} \beta^t u(H^{t+1}k_0, 1)$$

Since $F(k, L)$ is concave, assumption F2 implies

$$Mk_t \geq F(k_t, 1) \geq F(k_t, 1) - F(0, 1) \geq kF_k(0, 1) \geq k_t\delta.$$

Thus $M \geq \delta$ and $H = M + 1 - \delta \geq 1$. Note that if U is concave and $U(0) = 0$ then $U(Hx) \leq HU(x)$ for all $H \geq 1$. Denote $u[x, 1] = U(x)$. Then

$$\begin{aligned}
\sum_{t=T+1}^{\infty} \beta^t u(c_t, l_t) &\leq \sum_{t=T+1}^{\infty} \beta^t U(H^{t+1}k_0) \\
&\leq \sum_{t=T+1}^{\infty} (\beta H)^t U(Hk_0) \\
&\leq \varepsilon \text{ as } \beta H < 1 \text{ and } T \text{ is large enough.}
\end{aligned}$$

□

Denote $\mathbf{x} = \{\mathbf{c}, \mathbf{k}, \mathbf{l}\}$ and $\mathcal{F}(\mathbf{x}) = -\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$, $\Phi_t^1(\mathbf{x}) = c_t + k_{t+1} - f(k_t, 1 - l_t)$, $\Phi_t^2(\mathbf{x}) = -c_t$, $\Phi_t^3(\mathbf{x}) = -k_t$, $\Phi_t^4(\mathbf{x}) = -l_t$, $\Phi_t^5(\mathbf{x}) = l_t - 1$, $\forall t, \Phi_t = (\Phi_t^1, \Phi_t^2, \Phi_{t+1}^3, \Phi_t^4, \Phi_t^5)$, $\forall t$. The planning problem can be written as:

$$\min \mathcal{F}(\mathbf{x}) \quad \text{s.t. } \Phi(\mathbf{x}) \leq \mathbf{0}, \mathbf{x} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty \qquad (P)$$

$$\begin{aligned}
\text{where } \mathcal{F} &: \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty \to \mathbb{R} \cup \{+\infty\} \\
\Phi &= (\Phi_t)_{t=0,\ldots,\infty} : \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty \to \mathbb{R} \cup \{+\infty\}
\end{aligned}$$

$$\begin{aligned}\text{Let } C &= dom(\mathcal{F}) = \{\mathbf{x} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty | \mathcal{F}(\mathbf{x}) < +\infty\} \\ \Gamma &= dom(\Phi) = \{\mathbf{x} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty | \Phi_t(\mathbf{x}) < +\infty,\ \forall t\}.\end{aligned}$$

The following two propositions are an extension of Le Van and Saglam (2004) to the case of the optimal growth model with endogenous labor-leisure choice. Using Lemma 1 and Lemma 2 above, the proofs are sketched as they extend from the results in that paper. The propositions are given as they are crucial for the main results of our paper, which is in the next section.

Proposition 1. *Let* $\mathbf{x}, \mathbf{y} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty, T \in \mathbb{N}$. *Define*

$$x_t^T(\mathbf{x}, \mathbf{y}) = \begin{cases} x_t & if \quad t \leq T \\ y_t & if \quad t > T \end{cases}.$$

Suppose that the two following assumptions are satisfied:

T1: *If* $\mathbf{x} \in C$, $\mathbf{y} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$ *satisfy* $\forall T \geq T_0$, $\mathbf{x}^T(\mathbf{x}, \mathbf{y}) \in C$, *then* $\mathcal{F}(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) \to \mathcal{F}(\mathbf{x})$ *when* $T \to \infty$.

T2: *If* $\mathbf{x} \in \Gamma$, $\mathbf{y} \in \Gamma$ *and* $\mathbf{x}^T(\mathbf{x}, \mathbf{y}) = (x_t^T(\mathbf{x}, \mathbf{y}))_{t=0,1..\infty} \in \Gamma$, $\forall T \geq T_0$, *then*

a) $\Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) \to \Phi_t(\mathbf{x})$ *as* $T \to \infty$

b) $\exists N$ *s.t.* $\forall T \geq T_0$, $\|\Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y}))\| \leq N$

c) $\forall T \geq T_0$, $\lim_{t\to\infty}[\Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) - \Phi_t(\mathbf{y})] = 0$.

Let $\mathbf{x}^*$ *be a solution to* (P) *and* $\mathbf{x}^0 \in C$ *satisfies the Slater condition:*

$$\sup_t \Phi_t(\mathbf{x}^0) < 0.$$

Suppose $\mathbf{x}^T(\mathbf{x}^*, \mathbf{x}^0) \in C \cap \Gamma$. *Then, there exists* $\mathbf{\Lambda} \in \ell_+^1 \backslash \{0\}$ *such that*

$$\mathcal{F}(\mathbf{x}) + \mathbf{\Lambda}\Phi(\mathbf{x}) \geq \mathcal{F}(\mathbf{x}^*) + \Lambda\Phi(\mathbf{x}^*),\ \forall \mathbf{x} \in (C \cap \Gamma)$$

and $\Lambda\Phi(\mathbf{x}^*) = 0$.

Proof. It is easy to see that $\ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$ is isomorphic with ℓ_+^∞, since, for example, there exists an isomorphism

$$\Pi : \ell_+^\infty \to \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty,$$

$$\Pi(\mathbf{x}) = ((x_0, x_3, x_6, \ldots)(x_1, x_4, x_7, \ldots), (x_2, x_5, x_8, \ldots))$$

and

$$\Pi^{-1}(\mathbf{u}, \mathbf{v}, \mathbf{s}) = (u_0, v_0, s_0, u_1, v_1, s_1, u_2, v_2, s_2, \ldots).$$

Thus, there exists an isomorphism $\Pi' : (\ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty)' \to (\ell_+^\infty)'$. It follows from Theorem 1 in [14] that there exists $\overline{\mathbf{\Lambda}} \in (\ell^\infty_+ \times \ell^\infty_+ \times \ell^\infty_+)'$. Let $\Lambda = \Pi'(\overline{\mathbf{\Lambda}}) \in (\ell^\infty_+)'$. Then, the results are derived by the analogous arguments where a standard separation theorem used[6] as in the Theorem 2 in Le Van and Saglam (2004). □

Note that ***T1*** holds when F is continuous in the product topology. ***T2c*** is satisfied if there is asymptotically insensitivity, i.e. if x is changed only on a finitely many values the constraint value for large t does not change that much (Dechert 1982). ***T2c*** is the asymptotically non-anticipatory assumption and requires Φ to be nearly weak-* continuous (Dechert 1982). ***T2b*** holds when when $\Gamma = \text{dom}(\Phi) = \ell^\infty$ and Φ is continuous (see Dechert (1982) and Le Van and Saglam (2004)).

Proposition 2. *If* $\mathbf{x}^* = (\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*)$ *is a solution to the following problem*[7]*:*

$$\max_{c_t, l_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \tag{Q}$$

$$\begin{aligned} &s.t.\ c_t + k_{t+1} - f(k_t, 1 - l_t) \le 0, \\ &-c_t \le 0,\ -k_{t+1} \le 0,\ 0 \le l_t \le 1, \end{aligned}$$

then there exists $\lambda = (\lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5) \in \ell^1_+ \times \ell^1_+ \times \ell^1_+ \times \ell^1_+ \times \ell^1_+, \lambda \neq \mathbf{0}$ *such that:* $\forall \mathbf{x} =$

[6] As the Remark 6.1.1 in Le Van and Dana (2003), assumption $F_k(0, 1) \ge \delta$ is equivalent to the Adequacy Assumption in Bewley (1972) and this assumption is crucial to have equilibrium prices in ℓ^1_+ since it implies that the production set has an interior point. Subsequently, it allows using a separation theorem in the infinite dimensional space to obtain Lagrange multipliers.

[7] A solution exists following a standard argument which is sketched for completeness. Observe that the feasible set is in a fixed ball of ℓ^∞ which is weak*-(ℓ^∞, ℓ^1) compact. We show that the function $\sum_{t=0}^\infty \beta^t u(c_t, l_t)$ is continuous in this topology on the feasible set. Since the weak* topology is metrizable on any ball, we can take a feasible sequence $(c_t(n), l_t(n))_n$ converging to some (c_t, l_t) in the feasible set. Since any feasible consumptions sequence is uniformly bounded by a number depending only on k_0, for any $\epsilon > 0$ there exists T_0 such that for any $T \ge T_0$, for any n, we have

$$\sum_{t \ge T} \beta^t u(c_t(n), l_t(n)) \le \epsilon,\ \sum_{t \ge T} \beta^t u(c_t, l_t) \le \epsilon$$

Hence,

$$\left| \sum_{t=0}^{+\infty} \beta^t [u(c_t(n), l_t(n)) - u(c_t, l_t)] \right| \le \sum_{t=0}^{T-1} \beta^t |u(c_t(n), l_t(n)) - u(c_t, l_t)| + 2\epsilon.$$

Since weak* convergence implies pointwise convergence, the result is established.

$(\mathbf{c},\mathbf{k},\mathbf{l}) \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$

$$\begin{aligned}
& \sum_{t=0}^{\infty} \beta^t u(c_t^*, l_t^*) - \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* + k_{t+1}^* - f(k_t^*, 1-l_t^*)) \\
& + \sum_{t=0}^{\infty} \lambda_t^2 c_t^* + \sum_{t=0}^{\infty} \lambda_t^3 k_{t+1}^* + \sum_{t=0}^{\infty} \lambda_t^4 l_t^* + \sum_{t=0}^{\infty} \lambda_t^5 (1-l_t^*) \\
\geq \; & \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) - \sum_{t=0}^{\infty} \lambda_t^1 (c_t + k_{t+1} - f(k_t, 1-l_t)) \\
& + \sum_{t=0}^{\infty} \lambda_t^2 c_t + \sum_{t=0}^{\infty} \lambda_t^3 k_{t+1} + \sum_{t=0}^{\infty} \lambda_t^4 l_t + \sum_{t=0}^{\infty} \lambda_t^5 (1-l_t) \qquad (1)
\end{aligned}$$

$$\lambda_t^1 (c_t^* + k_{t+1}^* - f(k_t^*, 1-l_t^*)) = 0, \;\; \forall t \geq 0 \qquad (2)$$

$$\lambda_t^2 c_t^* = 0, \qquad (3)$$

$$\lambda_t^3 k_{t+1}^* = 0, \qquad (4)$$

$$\lambda_t^4 l_t^* = 0, \qquad (5)$$

$$\lambda_t^5 (1-l_t^*) = 0, \qquad (6)$$

$$0 \in \beta^t \partial_1 u(c_t^*, l_t^*) - \{\lambda_t^1\} + \{\lambda_t^2\}, \;\; \forall t \geq 0 \qquad (7)$$

$$0 \in \beta^t \partial_2 u(c_t^*, l_t^*) - \lambda_t^1 \partial_2 f(k_t^*, L_t^*) + \{\lambda_t^4\} - \{\lambda_t^5\}, \;\; \forall t \geq 0 \qquad (8)$$

$$0 \in \lambda_{t+1}^1 \partial_1 f(k_{t+1}^*, L_{t+1}^*) + \{\lambda_t^3\} - \{\lambda_t^1\}, \;\; \forall t \geq 0 \qquad (9)$$

where $\partial_i u(c_t^*, l_t^*), \partial_i f(k_t^*, L_t^*)$ *respectively denote the projection on the* i^{th} *component of the subdifferential of the function* u *at* (c_t^*, l_t^*) *and the function* f *at* (k_t^*, L_t^*).

Proof. We first check that the Slater condition holds. Assumption **F2** implies that

$$f_k(0,1) = F_k(0,1) + (1-\delta) > 1.$$

Then for all $k_0 > 0$, there exists some $0 < \widehat{k} < k_0$ such that: $0 < \widehat{k} < f(\widehat{k}, 1)$ and $0 < \widehat{k} < f(k_0, 1)$. Thus, there exists two small positive numbers ε, ε_1 such that:

$$0 < \widehat{k} + \varepsilon < f(\widehat{k}, 1-\varepsilon_1) \text{ and } 0 < \widehat{k} + \varepsilon < f(k_0, 1-\varepsilon_1).$$

Denote $\mathbf{x}^0 = (\mathbf{c}^0, \mathbf{k}^0, \mathbf{l}^0)$ such that $\mathbf{c}^0 = (\varepsilon, \varepsilon, ...)$, $\mathbf{k}^0 = (k_0, \widehat{k}, \widehat{k}, ...)$, $\mathbf{l}^0 = (\varepsilon_1, \varepsilon_1, ...)$. We have

$$\begin{aligned}
\Phi_0^1(\mathbf{x}^0) &= c_0 + k_1 - f(k_0, 1 - l_0) \\
&= \varepsilon + \widehat{k} - f(k_0, 1 - \varepsilon_1) < 0 \\
\Phi_1^1(\mathbf{x}^0) &= c_1 + k_2 - f(k_1, 1 - l_1) \\
&= \varepsilon + \widehat{k} - f(\widehat{k}, 1 - \varepsilon_1) < 0 \\
\Phi_t^1(\mathbf{x}^0) &= \varepsilon + \widehat{k} - f(\widehat{k}, 1 - \varepsilon_1) < 0, \ \forall t \geq 2
\end{aligned}$$

$$\Phi_t^2(\mathbf{x}^0) = -\varepsilon < 0, \ \forall t \geq 0, \ \ \Phi_0^3(\mathbf{x}^0) = -k_0 < 0$$

$$\Phi_t^3(\mathbf{x}^0) = -\widehat{k} < 0, \ \ \forall t \geq 1, \ \ \Phi_t^4(\mathbf{x}^0) = -\varepsilon_1 < 0, \ \forall t \geq 0$$

$$\Phi_t^5(\mathbf{x}^0) = \varepsilon_1 - 1 < 0, \ \forall t \geq 0.$$

Therefore, the Slater condition is satisfied.

We now check Assumption **T1**. For any $\widetilde{\mathbf{x}} \in C$, $\widetilde{\widetilde{\mathbf{x}}} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$ such that for any T, $\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}}) \in C$ we have

$$\mathcal{F}(\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}})) = -\sum_{t=0}^{T} \beta^t u(\widetilde{c}_t, \widetilde{l}_t) - \sum_{t=T+1}^{\infty} \beta^t u(\widetilde{\widetilde{c}}_t, \widetilde{\widetilde{l}}_t).$$

It follows from Lemma 1 $\sum_{t=T+1}^{\infty} \beta^t u(\widetilde{\widetilde{c}}_t, \widetilde{\widetilde{l}}_t) \to 0$ as $T \to \infty$. Hence, $\mathcal{F}(\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}})) \to \mathcal{F}(\widetilde{\mathbf{x}})$ when $T \to \infty$. Taking account of Proposition 1, we get (1)-(6).

Clearly, $\forall T$, $\mathbf{x}^T(\mathbf{x}^*, \mathbf{x}^0)$ belongs to $\ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$. Using Lemma**??**, Assumption T2 is satisfied as in Le Van and Saglam (2004).

Finally, we obtain (7)-(9) from the Kuhn-Tucker first-order conditions. □

3 Competitive equilibrium

Definition 1. *A competitive equilibrium consists of an allocation* $\{\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*, \mathbf{L}^*\} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$*, a price sequence* $\mathbf{p}^* \in \ell_+^1$ *for the consumption good, a wage sequence* $\mathbf{w}^* \in \ell_+^1$ *for labor and a price* $r > 0$ *for the initial capital stock* k_0 *such that:*

i) $\{\mathbf{c}^*, \mathbf{l}^*\}$ *is a solution to the problem*

$$\begin{aligned}
\max \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\
s.t. \quad & \mathbf{p}^* \mathbf{c} \leq \mathbf{w}^* \mathbf{L} + \pi^* + r k_0
\end{aligned}$$

where π^* *is the maximum profit of the firm.*

ii) $\{\mathbf{k}^*, \mathbf{L}^*\}$ *is a solution to the firm's problem*

$$\begin{aligned} \pi^* &= \max \sum_{t=0}^{\infty} p_t^*[f(k_t, L_t) - k_{t+1}] - \sum_{t=0}^{\infty} w_t^* L_t - rk_0 \\ s.t. \quad 0 &\leq k_{t+1} \leq f(k_t, L_t), \;\; L_t \geq 0, \forall t. \end{aligned}$$

iii) Markets clear

$$\begin{aligned} c_t^* + k_{t+1}^* &= f(k_t^*, L_t^*) \;\; \forall t \\ l_t^* + L_t^* &= 1 \;\; \forall t \\ \textit{and } k_0^* &= k_0 \end{aligned}$$

In the previous section, we show that under maintained assumptions, there exist multipliers that are summable. We now show that the appropriately chosen multipliers constitute a system of competitive equilibrium prices. The results on existence of a competitive equilibrium in the optimal growth model with inelastic labor supply do not extend immediately to the case of endogenous labor-leisure choice. The difficulty is that the previous results, e.g. Le Van and Vailakis (2004) rely on showing that the allocation is interior as the price of the good is the discounted marginal utility of consumption (See Remark 3 and Theorem 1 in that paper). As we show in the next section under our assumptions, a competitive equilibrium can exist even if the capital stock is zero or if the consumption is zero so that the price system in Le Van and Vailakis (2004) is not defined.

Theorem 1. *Let* $\{\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*\}$ *solve Problem* (Q)*. Take*

$$p_t^* = \lambda_t^1 \textit{ for any } t \textit{ and } r > 0.$$

There exists $f_L(k_t^*, L_t^*) \in \partial_2 f(k_t^*, L_t^*)$ *such that* $\{\mathbf{c}^*, \mathbf{k}^*, \mathbf{L}^*, \mathbf{p}^*, \mathbf{w}^*, r\}$ *is a competitive equilibrium with* $w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*)$*.*

Proof. Consider $\lambda = \{\lambda^{\mathbf{1}}, \lambda^{\mathbf{2}}, \; \lambda^{\mathbf{3}}, \lambda^{\mathbf{4}}, \lambda^{\mathbf{5}}\}$ of Proposition 2. Conditions (7), (8), (9) in Proposition 2 show that $\partial u(c_t^*, l_t^*)$ and $\partial f(k_t^*, L_t^*)$ are nonempty and there exist $u_c(c_t^*, l_t^*) \in \partial_1 u(c_t^*, l_t^*)$, $u_l(c_t^*, l_t^*) \in \partial_2 u(c_t^*, l_t^*)$, $f_k(k_t^*, L_t^*) \in \partial_1 f(k_t^*, L_t^*)$ and $f_L(k_t^*, L_t^*) \in \partial_2 f(k_t^*, L_t^*)$ such that $\forall t$

$$\beta^t u_c(c_t^*, l_t^*) - \lambda_t^1 + \lambda_t^2 = 0 \tag{10}$$

$$\beta^t u_l(c_t^*, l_t^*) - \lambda_t^1 f_L(k_t^*, L_t^*) + \lambda_t^4 - \lambda_t^5 = 0 \tag{11}$$

$$\lambda_{t+1}^1 f_k(k_{t+1}^*, L_{t+1}^*) + \lambda_t^3 - \lambda_t^1 = 0 \tag{12}$$

Define $w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*) < +\infty$.

First, we claim that $\mathbf{w}^* \in \ell^1_+$.

We have

$$+\infty > \sum_{t=0}^{\infty} \beta^t u(c_t^*, l_t^*) - \sum_{t=0}^{\infty} \beta^t u(0,0) \geq \sum_{t=0}^{\infty} \beta^t u_c(c_t^*, l_t^*) c_t^* + \sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*) l_t^*,$$

which implies

$$\sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*) l_t^* < +\infty, \tag{13}$$

and

$$+\infty > \sum_{t=0}^{\infty} \lambda_t^1 f(k_t^*, L_t^*) - \sum_{t=0}^{\infty} \lambda_t^1 f(0,0) \geq \sum_{t=0}^{\infty} \lambda_t^1 f_k(k_t^*, L_t^*) k_t^* + \sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^*$$

which implies

$$\sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* < +\infty. \tag{14}$$

Given T, we multiply (11) by L_t^* and sum up from 0 to T. Observe that

$$\forall T, \sum_{t=0}^{T} \beta^t u_l(c_t^*, l_t^*) L_t^* = \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* + \sum_{t=0}^{T} \lambda_t^5 L_t^* - \sum_{t=0}^{T} \lambda_t^4 L_t^*. \tag{15}$$

$$0 \leq \sum_{t=0}^{\infty} \lambda_t^5 L_t^* \leq \sum_{t=0}^{\infty} \lambda_t^5 < +\infty. \tag{16}$$

$$0 \leq \sum_{t=0}^{\infty} \lambda_t^4 L_t^* \leq \sum_{t=0}^{\infty} \lambda_t^4 < +\infty. \tag{17}$$

Thus, since $L_t^* = 1 - l_t^*$, from (15), we get

$$\begin{aligned}\sum_{t=0}^{T} \beta^t u_l(c_t^*, l_t^*) &= \sum_{t=0}^{T} \beta^t u_l(c_t^*, l_t^*) l_t^* + \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* \\ &+ \sum_{t=0}^{T} \lambda_t^5 L_t^* - \sum_{t=0}^{T} \lambda_t^4 L_t^*.\end{aligned}$$

Using (13), (14), (16), (17) and letting $T \to \infty$, we obtain

$$\begin{aligned} 0 &\le \sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*) = \sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*) l_t^* + \sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* \\ &+ \sum_{t=0}^{\infty} \lambda_t^5 L_t^* - \sum_{t=0}^{\infty} \lambda_t^4 L_t^* < +\infty. \end{aligned}$$

Consequently, from (11), $\sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) < +\infty$ i.e. $\mathbf{w}^* \in \ell_+^1$. So, we have $\{\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*, \mathbf{L}^*\} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$, with $\mathbf{p}^* \in \ell_+^1$ and $\mathbf{w}^* \in \ell_+^1$.

We now show that $(\mathbf{k}^*, \mathbf{L}^*)$ is solution to the firm's problem.

Since $p_t^* = \lambda_t^1$, $w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*)$, we have

$$\pi^* = \sum_{t=0}^{\infty} \lambda_t^1 [f(k_t^*, L_t^*) - k_{t+1}^*] - \sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*)\ L_t^* - r k_0$$

Let :

$$\begin{aligned} \Delta_T &= \sum_{t=0}^{T} \lambda_t^1 [f(k_t^*, L_t^*) - k_{t+1}^*] - \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*)\ L_t^* - r k_0 \\ &- \left(\sum_{t=0}^{T} \lambda_t^1 [f(k_t, L_t) - k_{t+1}] - \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*)\ L_t - r k_0 \right). \end{aligned}$$

From the concavity of f, we get

$$\begin{aligned} \Delta_T &\ge \sum_{t=1}^{T} \lambda_t^1 f_k(k_t^*, L_t^*)(k_t^* - k_t) - \sum_{t=0}^{T} \lambda_t^1 (k_{t+1}^* - k_{t+1}) \\ &= [\lambda_1^1 f_k(k_1^*, L_1^*) - \lambda_0^1](k_1^* - k_1) + \dots \\ &\quad + [\lambda_T^1 f_k(k_T^*, L_T^*) - \lambda_{T-1}^1](k_T^* - k_T) \\ &\quad - \lambda_T^1 (k_{T+1}^* - k_{T+1}). \end{aligned}$$

By (4) and (12), we have: $\forall t = 1, 2, \dots, T$

$$[\lambda_t^1 f_k(k_t^*, L_t^*) - \lambda_{t-1}^1](k_t^* - k_t) = -\lambda_{t-1}^3 (k_t^* - k_t) \ge -\lambda_{t-1}^3 k_t^*.$$

Thus,

$$\Delta_T \ge -\sum_{t=1}^{T} \lambda_{t-1}^3 k_t^* - \lambda_T^1 (k_{T+1}^* - k_{T+1}) \ge -\sum_{t=1}^{T} \lambda_{t-1}^3 k_t^* - \lambda_T^1 k_{T+1}^*.$$

Since $\lambda^1, \lambda^3 \in \ell^1_+, \sup\limits_T \frac{k^*_{T+1}}{H^{T+1}} < +\infty$, we have

$$\lim_{T\to+\infty} \Delta_T \geq 0.$$

We have proved that the sequences $(\mathbf{k}^*, \mathbf{L}^*)$ maximize the profit of the firm. We now show that $\mathbf{c}^*$ solves the consumer's problem.

$$\text{Let } \{\mathbf{c}, \mathbf{L}\} \text{ satisfy } \sum_{t=0}^{\infty} \lambda_t^1 c_t \leq \sum_{t=0}^{\infty} w_t^* L_t + \pi^* + rk_0. \tag{18}$$

By the concavity of u, we have:

$$\Delta = \sum_{t=0}^{\infty} \beta^t u(c_t^*, l_t^*) - \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$\geq \sum_{t=0}^{\infty} \beta^t u_c(c_t^*, l_t^*)(c_t^* - c_t) + \sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*)\ (l_t^* - l_t).$$

Combining (3), (6), (10), (11) yields

$$\Delta \geq \sum_{t=0}^{\infty} (\lambda_t^1 - \lambda_t^2)(c_t^* - c_t) + \sum_{t=0}^{\infty} (\lambda_t^1 f_L(k_t^*, 1 - l_t^*) + \lambda_t^5 - \lambda_t^4)(l_t^* - l_t)$$

$$= \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* - c_t) + \sum_{t=0}^{\infty} \lambda_t^2 c_t - \sum_{t=0}^{\infty} \lambda_t^2 c_t^* + \sum_{t=0}^{\infty} (w_t^* + \lambda_t^5)(l_t^* - l_t)$$

$$- \sum_{t=0}^{\infty} \lambda_t^4 l_t^* + \sum_{t=0}^{\infty} \lambda_t^4 l_t$$

$$\begin{aligned} \geq\ & \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* - c_t) + \sum_{t=0}^{\infty} (w_t^* + \lambda_t^5)(l_t^* - l_t) = \\ & \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* - c_t) + \sum_{t=0}^{\infty} w_t^* (l_t^* - l_t) + \sum_{t=0}^{\infty} \lambda_t^5 (1 - l_t) \\ \geq\ & \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* - c_t) + \sum_{t=0}^{\infty} w_t^* (L_t - L_t^*). \end{aligned}$$

Since

$$\pi^* = \sum_{t=0}^{\infty} \lambda_t^1 c_t^* - \sum_{t=0}^{\infty} w_t^* L_t^* - rk_0,$$

it follows from (18) that

$$\begin{aligned}\Delta &\geq \sum_{t=0}^{\infty} p_t^* c_t^* - \sum_{t=0}^{\infty} w_t^* L_t^* - rk_0 - \left(\sum_{t=0}^{\infty} p_t^* c_t - \sum_{t=0}^{\infty} w_t^* L_t - rk_0 \right) \\ &\geq \pi^* - \pi^* = 0\end{aligned}$$

Consequently, $\Delta \geq 0$ that means c^* solves the consumer's problem.

Finally, the market clears at every period, since $\forall t, c_t^* + k_{t+1}^* = f(k_t^*, L_t^*)$ and $1 - l_t^* = L_t^*$. □

4 Examples

We give three parametric example illustrating generality of our result. In these examples there are corner solutions that the literature makes assumptions to rule out. In each of the examples the competitive equilibrium is calculated. They illustrate that the interiority of an allocation is not necessary for existence of a competitive equilibrium.

In the first example, there is a competitive equilibrium with zero labor supply, the good being produced through capital alone. As a consumer may choose to enjoy all available time as leisure, imposing an Inada condition on productivity of labor is not well justified. As we show in this case we still have existence of competitive equilibria. In the second example, we show that a competitive equilibrium will exist even if $k_t = 0$, $\forall t$, thus, showing that the capital stock is positive is not necessary for existence. In this example, the good is produced through labor alone, and consumption of both the good and leisure is positive. The third example, shows that it can be the case that $c_t = L_t = 0$, that is the consumer just consumes leisure. Furthermore, all capital is reinvested and can be unbounded. In this case, the price system as in Le Van and Vailakis (2004) is not defined.

4.1 Example 1: Competitive equilibrium with $L_t^* = 0, l_t^* = 1$

Consider an economy with a good that can either be consumed or invested as capital, one firm and one consumer. The consumer has preferences defined over processes of consumption and leisure described by the utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t) = \sum_{t=0}^{\infty} \beta^t (c_t + ml_t),$$

The firm produces capital good by using capital k_t and labor $L_t = 1 - l_t$. The production function $f(k_t, L_t) = (k_t^\alpha + L_t)^{1/\theta}, 0 < \alpha < \theta, 0 < \beta < 1 < \theta, f$ is concave and increasing.

Assume that $m = \frac{1}{\theta}\left(\frac{\theta}{\beta\alpha}\right)^{\dfrac{\alpha(\theta-1)}{\theta-\alpha}}$. The planning problem is

$$
\begin{aligned}
& \max_{c_t, l_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t (c_t + m l_t) \\
\text{s.t. } c_t + k_{t+1} &\leq (k_t^\alpha + L_t)^{1/\theta}, \ \forall t \geq 0 \\
L_t + l_t &= 1, \ \forall t \geq 0 \\
c_t &\geq 0, k_{t+1} \geq 0, l_t \geq 0, 1 - l_t \geq 0, \ \forall t \geq 0 \\
k_0 &\geq 0 \text{ is given.}
\end{aligned}
$$

Inada conditions are not satisfied for both the utility and production functions.[8] The utility function is also not strictly concave.

Let $\lambda_t = (\lambda_t^i)_{i=1}^5, \lambda_t \neq 0$ denote the Lagrange multipliers. The Lagrangean is

$$
\begin{aligned}
\mathcal{H} &= \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) - \sum_{t=0}^{\infty} \lambda_t^1 (c_t + k_{t+1} - f(k_t, 1 - l_t)) \\
&\quad + \sum_{t=0}^{\infty} \lambda_t^2 c_t + \sum_{t=0}^{\infty} \lambda_t^3 k_{t+1} + \sum_{t=0}^{\infty} \lambda_t^4 l_t + \sum_{t=0}^{\infty} \lambda_t^5 (1 - l_t)
\end{aligned}
$$

It follows from Kuhn-Tucker necessary conditions that, $\forall t \geq 0$

$$
\begin{aligned}
0 &= \beta^t - \lambda_t^1 + \lambda_t^2 \\
0 &= \beta^t m - \frac{1}{\theta} \lambda_t^1 (k_t^\alpha + 1 - l_t)^{\frac{1-\theta}{\theta}} + \lambda_t^4 - \lambda_t^5 \\
0 &= \frac{\alpha}{\theta} \lambda_{t+1}^1 k_{t+1}^{\alpha-1} (k_{t+1}^\alpha + 1 - l_{t+1})^{\frac{1-\theta}{\theta}} + \lambda_t^3 - \lambda_t^1 \\
0 &= \lambda_t^1 (c_t + k_{t+1} - (k_t^\alpha + L_t)^{1/\theta}) \\
\lambda_t^2 c_t &= \lambda_t^3 k_{t+1} = \lambda_t^4 l_t = \lambda_t^5 (1 - l_t) = 0.
\end{aligned}
$$

[8]Linearity of the utility function in consumption is not important for the two examples. This, however, simplifies the calculations.

It is easy to check that, the above system of equation has a solution $\forall t \geq 1$:

$$\begin{aligned}
\lambda_t^{*1} &= \beta^t, \lambda_t^{*2} = \lambda_t^{*3} = \lambda_t^{*4} = \lambda_t^{*5} = 0, \\
k_t^* &= (\frac{\beta\alpha}{\theta})^{\frac{\theta}{\theta-\alpha}} := k_s \in (0,1), \\
c_t^* &= (k_s)^{\alpha/\theta} - k_s > 0, \\
l_t^* &= 1, \\
L_t^* &= 0.
\end{aligned}$$

As we show in section 3, if we define the sequence price $p_t^* = \lambda_t^{*1} = \beta^t$ for the consumption good and $w_t^* \in \lambda_t^{*1}\partial_2 f(k_t^*, L_t^*) = \lambda_t^{*1} f_L(k_s, 0) = \beta^t \frac{1}{\theta} k_s^{\frac{\alpha(1-\theta)}{\theta}}$ then $p_t^* \in \ell_+^1$, $w_t^* \in \ell_+^1$ and $\{\mathbf{c}^*, \mathbf{k}^*, \mathbf{L}^*, \mathbf{p}^*, \mathbf{w}^*, r\}$ is a competitive equilibrium.

4.2 Example 2: Competitive equilibrium with $k_t^* = 0$

Now consider the production function $f(k_t, L_t) = (k_t + L_t^\alpha)^{1/\theta}$ where $0 < \alpha < \theta, 0 < \beta < 1 < \theta$ and the utility function

$$u(c_t, l_t) = c_t + \frac{1}{\theta}\left(\frac{\beta}{\theta}\right)^{\frac{\alpha-\theta}{\alpha(\theta-1)}} l_t$$

We obtain the Kuhn-Tucker conditions, $\forall t \geq 0$

$$\begin{aligned}
0 &= \beta^t - \lambda_t^1 + \lambda_t^2 \\
0 &= \frac{1}{\theta}(\frac{\beta}{\theta})^{\frac{\alpha-\theta}{\alpha(\theta-1)}}\beta^t - \frac{\alpha}{\theta}\lambda_t^1 L_t{}^{\alpha-1}(k_t + L_t^\alpha)^{\frac{1-\theta}{\theta}} + \lambda_t^4 - \lambda_t^5 \\
0 &= \frac{1}{\theta}\lambda_{t+1}^1(k_{t+1} + L_{t+1}^\alpha)^{\frac{1-\theta}{\theta}} + \lambda_t^3 - \lambda_t^1 \\
0 &= \lambda_t^1(c_t + k_{t+1} - (k_t + L_t^\alpha)^{1/\theta}) \\
\lambda_t^2 c_t &= \lambda_t^3 k_{t+1} = \lambda_t^4 l_t = \lambda_t^5(1 - l_t) = 0.
\end{aligned}$$

The system of equation has solution $\forall t \geq 1$:

$$\begin{aligned}
\lambda_t^{*1} &= \beta^t, \lambda_t^{*2} = \lambda_t^{*3} = \lambda_t^{*4} = \lambda_t^{*5} = 0 \\
k_t^* &= 0 \\
L_t^* &= (\frac{\beta}{\theta})^{\frac{\theta}{\alpha(\theta-1)}} := L_s \in (0,1) \\
c_t^* &= (L_s)^{\alpha/\theta} > 0 \\
l_t^* &= 1 - L_s.
\end{aligned}$$

4.3 Example 3: Competitive equilibrium with $L_t^* = c_t^* = 0, l_t^* = 1$

Consider an economy with a good that can either be consumed or invested as capital, one firm and one consumer. The consumer has preferences defined over processes of consumption and leisure described by the utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t) = \sum_{t=0}^{\infty} \beta^t (c_t + m l_t),$$

The firm produces capital good by using capital k_t and labor $L_t = 1 - l_t$. The production function $f(k_t, L_t) = Mk_t + mL_t, 0 < m$.

Let choose $M = \frac{1}{\beta} - a$ where $a > 1$.In this example, depreciation rate $\delta = 0$, assumption F3 is satisfied as $M = \frac{1}{\beta} - a < \frac{1}{\beta} - 1 + \delta$.

Inada conditions are not satisfied for both the utility and production functions.

The planning problem is

$$\begin{aligned}
& \max_{c_t, l_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t (c_t + m l_t) \\
\text{s.t. } c_t + k_{t+1} &\leq Mk_t + mL_t, \ \forall t \geq 0 \\
L_t + l_t &= 1, \ \forall t \geq 0 \\
c_t &\geq 0, k_{t+1} \geq 0, l_t \geq 0, 1 - l_t \geq 0, \ \forall t \geq 0 \\
k_0 &\geq 0 \text{ is given.}
\end{aligned}$$

Let $\lambda_t = (\lambda_t^i)_{i=1}^5, \lambda_t \neq 0$ denote the Lagrange multipliers. The Lagrangean is

$$\begin{aligned}
\mathcal{H} &= \sum_{t=0}^{\infty} \beta^t (c_t + m l_t) - \sum_{t=0}^{\infty} \lambda_t^1 [c_t + k_{t+1} - Mk_t - m(1 - l_t)] \\
&+ \sum_{t=0}^{\infty} \lambda_t^2 c_t + \sum_{t=0}^{\infty} \lambda_t^3 k_{t+1} + \sum_{t=0}^{\infty} \lambda_t^4 l_t + \sum_{t=0}^{\infty} \lambda_t^5 (1 - l_t).
\end{aligned}$$

It follows from Kuhn-Tucker necessary conditions that, $\forall t \geq 0$

$$\begin{aligned}
0 &= \beta^t - \lambda_t^1 + \lambda_t^2 \\
0 &= \beta^t m - \lambda_t^1 m + \lambda_t^4 - \lambda_t^5 \\
0 &= M\lambda_{t+1}^1 - \lambda_t^1 + \lambda_t^3 \\
0 &= \lambda_t^1 [c_t + k_{t+1} - Mk_t - m(1 - l_t)] \\
\lambda_t^2 c_t &= \lambda_t^3 k_{t+1} = \lambda_t^4 l_t = \lambda_t^5 (1 - l_t) = 0.
\end{aligned}$$

It is easy to check that, the above system of equation has a solution :

$$\begin{aligned}
\lambda_t^{*1} &= \beta^t, \forall t \geq 0\\
\lambda_t^{*2} &= \lambda_t^{*4} = \lambda_t^{*5} = 0, \forall t \geq 0\\
\lambda_t^{*3} &= a\beta^{t+1}\\
c_t^* &= 0, \forall t \geq 0\\
k_0^* &= k_0, k_t^* = Mk_{t-1}^* = M^t k_0, \forall t \geq 1\\
L_t^* &= 0, \forall t \geq 0\\
l_t^* &= 1, \forall t \geq 0.
\end{aligned}$$

We prove that this solution is indeed the optimal solution of social planer problem. Let us consider

$$\begin{aligned}
\Delta_T &= \sum_{t=0}^{T} \beta^t (c_t^* + ml_t^*) - \sum_{t=0}^{T} \beta^t (c_t + ml_t)\\
&= \sum_{t=0}^{T} \beta^t [\frac{1}{\beta}(k_t^* - k_t) - (k_{t+1}^* - k_{t+1}) + m(1 - l_t)]\\
&\geq \sum_{t=0}^{T} \beta^t [\frac{1}{\beta}(k_t^* - k_t) - (k_{t+1}^* - k_{t+1})]\\
&= \frac{1}{\beta}(k_0^* - k_0) + \sum_{t=1}^{T} \beta^{t-1}(k_t^* - k_t) - \sum_{t=0}^{T} \beta^t [k_{t+1}^* - k_{t+1}]\\
&= -\beta^T (k_{T+1}^* - k_{T+1}) \geq -\beta^T k_{T+1}^* = -M(\beta M)^T k_0.
\end{aligned}$$

As $T \to \infty, \lim_{T\to\infty} \Delta_T \geq \lim_{T\to\infty} -M(\beta M)^T k_0 = 0$. Or $\sum_{t=0}^{\infty} \beta^t (c_t^* + ml_t^*) \geq \sum_{t=0}^{\infty} \beta^t (c_t + ml_t)$. This implies $(c_t^*, k_t^*, L_t^*, l_t^*)$ solves the max planning problem. As we show in section 3, if we define the sequence price $p_t^* = \beta^t$ for the consumption good and wage $w_t^* = m\beta^t$ then $\{\mathbf{c}^*, \mathbf{k}^*, \mathbf{L}^*, \mathbf{p}^*, \mathbf{w}^*\}$ is a competitive equilibrium.

If we choose $1 < a < \frac{1-\beta}{\beta}$ then $\lim_{t\to\infty} k_t = +\infty$ (violates boundedness of the capital stock in literature) we still prove the existence of competitive equilibrium.

5 Discussion and Conclusion

This paper studies existence of equilibrium in the optimal growth model with elastic labor supply. This model is the workhorse of dynamic general equilibrium theory for both endogenous and real business cycles. The results on existence of equilibrium have assumed strong conditions which are violated in some specifications of applied models.

This paper uses a separation argument to obtain Lagrange multipliers which lie in ℓ^1. As the separation argument relies on convexity, strict convexity can be relaxed; this also means that assumptions on cross partials of utility functions are not needed (as in Aiyagari, et al. (1992), Coleman (1997), Datta, et al. (2002), Greenwood and Huffman (1995) and Le Van, et al. (2007)); and homogeneity of production is not needed. These above papers assume normality of leisure (rule out backward bending labor supply curves) to show that the capital path is monotonic but this is inessential to show existence of a competitive equilibrium. The representation theorem involves assumptions on asymptotic properties of the constraint set (which are weaker than Mackey continuity (see Bewley (1972) and Dechert (1982)). The assumptions ensure that the either the optimal sequence $\{c_t, l_t\}_{t=0}^{\infty}$ is either always strictly interior or always equal to zero. Thus, one does not have to impose strong conditions, either Inada conditions (see for example, Goenka, et al. 2011), or $\lim_{\epsilon\to 0} \dfrac{u(\epsilon,\epsilon)}{\epsilon} \to +\infty$ as in Le Van and Vailakis (2004) to ensure that the sequence of labor is strictly interior. This later condition is not satisfied, for example, in homogeneous period one utility functions. The existence result also does not employ any differentiability assumptions. Thus, it covers both Leontief utility and production functions $Y = min(K/v, L/u)$ and $Y/L = (1/v)K/L$. This implies that the intensive production function, $y = f(k)$ where $y = Y/L$ and $k = K/L$ is effectively a straight line with slope $1/v$ up to the capital-labor ratio $k^* = K^*/L^*$ and is horizontal thereafter. Another well known model where differentiability is violated is the Intensive Activity Analysis Production Function but existence follows from our results.

A careful reader will observe that we can introduce tax and other distortions for the existence of a competitive equilibrium as long as concavity is maintained in line with the results in the literature. For monotonicity results, stronger results need to be imposed.

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