Overlapping generations economy, environmental externalities, and taxation

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Abstract

I set up in this paper an overlapping generations economy with environment degrading itself and pollution resulting from both consumption and production to show that there always exists an inter-temporal equilibrium and to determine the competitive steady state. This steady state is compared with the equilibrium steady state in the social benevolent planner's point of view. The paper shows the optimal "golden rule" allocation which maximizes the total utility of all generations, and whenever the capital ratio in the competitive framework is higher than the golden rule capital ratio, the economy stands on the dynamically inefficient point. The width of the inefficient range of capital ratio depends positively on the environment maintaining technology and depends negatively on the cleanness of production technology. For such any competitive economy, I introduce some combinations of taxes and transfer with purpose of decentralizing the best steady state attainable through the good and factors markets.

Keywords: overlapping generations, environmental externality, taxes and transfer scheme.

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1 Introduction

Considerations on environmental externalities in the Overlapping Generations (OLG) framework have been taken into account since

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1990s. Most studies look at the effects of environment externalities on dynamic inefficiency, productivity, health and longevity of agents, as well as the desirable interventions of social authorities. Most papers take into account that pollution comes from the production process and that environment may recover or degrade itself at a constant rate (Marini and Scaramozzino 1995; Jouvet et al 2000; Jouvet, Pestieau and Ponthiere 2007; Pautrel 2007; Gutiérrez 2008). However, some other researchers assume that pollution comes only from consumption (John and Pecchenino 1994; John et al. 1995; Ono 1996). One interesting remark from the literature is that due to opposite assumptions in different models, the findings of effects of environmental externalities on capital accumulation are different across papers. John et al. (1995) showed that when only consumption degrades environment, the economy accumulates less capital than what would be optimal; meanwhile Gutiérrez (2008) showed that when production causes a higher pollution, the economy accumulates instead more capital than the optimal level. This is so because in John et al.'s model agents have to pay taxes to maintain environment when young, therefore an increase in pollution reduces their saving for the future; however, in Gutiérrez's model, higher environmental pollution increases health costs, which are paid in the old age, leads to agents have to save more. So the difference seems to come from when the taxes are paid (young or old?) rather than from whether it is production or consumption that pollutes. Another difference to be noted in these two papers is their different assumptions about the ability of environment to recover from pollution. John et al. (1995) assumed that environment degrades itself overtime meanwhile Gutiérrez assumed on the contrary that it improves by a self-purification process. It would be interesting to know then which difference follows from which assumption.

This paper tries to disentangle the effects of both production and consumption on environment simultaneously. As in John et al. (1994, 1995), we also assume that the environment degrades itself over time with a constant rate and the young agents spend an amount from their income in order to maintain environment. Moreover, in John et al.'s papers, they assume that only the consumption of old agents degrades the environment and young agents do not consume, while in the paper of Ono (1996), he assumes consumption of both young and old agents degrade the environment.

However, in his paper, the current consumptions do not degrade the current environment but they degrade the environment in the next period onward. Here, we also assume that the environment is degraded by the consumptions of both old and young agents but we assume instead that the environment is not only degraded by the past consumptions and production but also by the current consumptions and current production. Specifically, we characterize the dynamically inefficient range of capital ratios in the presence of environmental externalities. Then, we shall introduce some taxes and transfer policies that make the competitive equilibrium steady state to be efficient.

The rest of this paper is organized as follows. Section 2 introduces the model and define the competitive equilibrium and the competitive steady state. Section 3 presents the problem of the social planner and define the efficient allocation and optimal allocation; and we show the dynamically inefficient range of the capital ratio in the competitive framework (proposition 1). We will compare the competitive steady state and the social planner's steady state in section 4, and hence, introduce some taxes and transfer schemes to decentralize the best steady state through goods and factor markets. Section 5 concludes the paper.

2 The model and competitive equilibria

We consider the overlapping generations economy in Diamond (1965) without population growth (growth rate $n = 0$) and normalise the size of each generation to unity. Each agent in the economy lives two periods, say young and old respectively. When young, an agent is endowed with one unit of labor which he supplies to the producing firms inelastically. He divides his wage, w_t, between consumption when young c_t^t, investment in maintaining environment m_t, and savings k_{t+1} which will be consumed when old. He supplies his savings inelastically to producing firms and earns the gross return $r_{t+1}k_{t+1}$ to consume when old, where r_{t+1} is the rental rate of capital in the period $t + 1$. Agents born at date t have preferences defined over their consumptions in young and old ages $(c_t^t, c_{t+1}^t) \in \mathbb{R}_+^2$ and the index of the environmental quality, $E_{t+1} \in \mathbb{R}$, which they experience when old. The preference is represented by the utility function $U : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}$. We assume that

$U(c_t^t, c_{t+1}^t, E_{t+1}) = u(c_t^t) + v(c_{t+1}^t) + \phi(E_{t+1})$ is additively separable and that $U_i(\cdot) > 0$, $U_{ii}(\cdot) < 0$, $i \in \left\{c_t^t, c_{t+1}^t, E_{t+1}\right\}$.

Environmental quality evolves according to

$$E_{t+1} = (1-b)E_t - \alpha F(K_{t+1}, L_{t+1}) - \beta(c_{t+1}^{t+1} + c_{t+1}^t) + \gamma m_t$$

for some $\alpha, \beta, \gamma > 0$ and $b \in (0,1]$, where $F(\cdot,\cdot)$ is the production function of the economy.

We assume that there is only one production sector in the economy using two factors of production as capital K and labor L. In each period t, firms produce a quantity of output Y_t through a constant returns to scale Cobb-Douglas production function, $Y_t = F(K_t, L_t) = AK_t^{\theta} L_t^{1-\theta}$, and capital fully depreciates each period. In the framework of perfect competition, the representative profit maximizing firm chooses K_t and L_t to maximize its profit

$$\pi_t = \underset{K_t, L_t \geq 0}{Max} F(K_t, L_t) - r_t K_t - w_t L_t$$

so that in each period t, the wage rate and the rental rate of capital are determined respectively by the marginal productivity of labor and capital. Since at equilibrium the capital K_t available at any period t, given population is normalized at 1, is the previous aggregate savings k_t and aggregate labor is L_t, then the wage rate and rental rate of capital that the agent living in period t and $t+1$ faces are

$$r_{t+1} = F_K(K_{t+1}, L_{t+1}) = F_K(k_{t+1}, 1) = \theta A k_{t+1}^{\theta-1} \tag{1}$$

$$w_t = F_L(K_t, L_t) = F_L(k_t, 1) = (1-\theta)Ak_t^{\theta} \tag{2}$$

Without human activity, the environmental quality will converge autonomously to the level of zero and the depreciation rate b measures the speed of reversion to this level. The terms $\alpha F(K_{t+1}, 1)$ and $\beta(c_{t+1}^{t+1} + c_{t+1}^t)$ are the degradations of the environment quality resulting from production and consumption, respectively. The term γm_t measures environmental improvement from the action of young agents at period t. One can thus interpret environmental quality as the cleanness of rivers and atmosphere, and the quality of soil or groundwater, etc. It is also the quality of parks, gardens and zoos which themselves depreciate and also require maintenance. One example for an output of good that both producing it and consuming it degrade the environment is wood. People produce wood by cutting trees in the forests degrading the environment. If wood is used to maintain the parks or zoos then the environment is improved. But if consumed as fuel (for heating or domestic uses), then the environment will be polluted. Since we normalize the size of each generation to unity and know that the savings of agent born in the period t are used as capital for production in period $t+1$, then the evolution of the environmental quality can be represented by the following expression

$$E_{t+1} = (1-b)E_t - \alpha F(k_{t+1}, 1) - \beta(c_{t+1}^{t+1} + c_{t+1}^t) + \gamma m_t$$

where $F(k_{t+1}, 1)$ is the production function per capital in period $t+1$.

Formally, the life-time utility maximization problem of the representative agent is as follows

$$\underset{\substack{c_t^t, c_{t+1}^t, k_{t+1}, m_t \geq 0 \\ E_t,\, E_{t+1}^e}}{Max} u(c_t^t) + v(c_{t+1}^t) + \phi(E_{t+1}^e) \tag{3}$$

subject to

$$w_t = c_t^t + k_{t+1} + m_t \tag{4}$$

$$c_{t+1}^t = r_{t+1}k_{t+1} \tag{5}$$

$$E_t = (1-b)E_{t-1} - \alpha F(k_t, 1) - \beta(c_t^t + c_t^{t-1}) + \gamma m_{t-1} \tag{6}$$

$$E_{t+1}^e = (1-b)E_t - \alpha F(k_{t+1}, 1) - \beta(c_{t+1}^{t+1,e} + c_{t+1}^t) + \gamma m_t \tag{7}$$

for given values of E_{t-1}, c_t^{t-1}, k_t, m_{t-1}, w_t as well as the expected consumption of the young agent, $c_{t+1}^{t+1,e}$, and the rate of capital return, r_{t+1}, in the period $t+1$. We assume that this agent is one of the many members of generation t and therefore his savings are very small compared to the aggregate savings of the economy as a whole. As a consequence, he ignores the impact on the aggregate capital of the economy from his own savings. This assumption implies that he does not internalize the impact of his savings have on the environment via production. The first order conditions (FOCs) for the agent's problem are the following

$$u'(c_t^t) - [\beta(1-b)+\gamma]\,\phi'(E_{t+1}^e) = 0 \tag{8}$$

$$r_{t+1}v'(c_{t+1}^t) - [\beta r_{t+1}+\gamma]\,\phi'(E_{t+1}^e) = 0 \tag{9}$$

which relate the marginal utilities of consumptions with marginal utility of environmental quality. The optimal choice of the agent, $\{c_t^t,\, c_{t+1}^t,\, k_{t+1},\, m_t,\, E_t,\, E_{t+1}^e\}$, is a function of E_{t-1}, c_t^{t-1}, k_t, m_{t-1}, $c_{t+1}^{t+1,e}$, and r_{t+1} implicitly defined by the system of equations

$$c_t^t + k_{t+1} + m_t - w_t = 0 \tag{10}$$

$$c_{t+1}^t - r_{t+1}k_{t+1} = 0 \tag{11}$$

$$E_t - (1-b)E_{t-1} + \alpha F(k_t,1) + \beta(c_t^t + c_t^{t-1}) - \gamma m_{t-1} = 0 \tag{12}$$

$$E_{t+1}^e - (1-b)E_t + \alpha F(k_{t+1},1) + \beta(c_{t+1}^{t+1,e} + c_{t+1}^t) - \gamma m_t = 0 \tag{13}$$

$$u'(c_t^t) - [\beta(1-b)+\gamma]\,\phi'(E_{t+1}^e) = 0 \tag{14}$$

$$v'(c_{t+1}^t) - \left[\beta + \frac{\gamma}{r_{t+1}}\right]\phi'(E_{t+1}^e) = 0 \tag{15}$$

as long as the Jacobian matrix of the left-hand-side of the system above with respect to c_t^t, c_{t+1}^t, k_{t+1}, m_t, E_t, E_{t+1}^e is regular at the solution. The regularity of the associated Jacobian matrix will be verified at the equilibrium in Appendix A1.

In order to guarantee the FOCs are not only necessary but sufficient for the solution to be a maximum, we have check the second order conditions (SOCs) which are represented in the Appendix A2.

2.1 Competitive equilibrium

The perfect foresight competitive equilibrium allocations are characterized by the agent maximizing utility under these budget constraints holding correct expectations, the dynamics of environment, and the determinants of the factors' prices. In other words, they are the solution to the following system of equations

$$c_t^t + k_{t+1} + m_t - F_L(k_t, 1) = 0 \tag{16}$$

$$c_{t+1}^t - F_K(k_{t+1}, 1)k_{t+1} = 0 \tag{17}$$

$$E_{t+1} - (1-b)E_t + \alpha F(k_{t+1}, 1) + \beta(c_{t+1}^{t+1} + c_{t+1}^t) - \gamma m_t = 0 \tag{18}$$

$$u'(c_t^t) - [\beta(1-b) + \gamma]\, \phi'(E_{t+1}) = 0 \tag{19}$$

$$v'(c_{t+1}^t) - \left[\beta + \frac{\gamma}{F_K(k_{t+1}, 1)}\right] \phi'(E_{t+1}) = 0 \tag{20}$$

The existence of perfect foresight competitive equilibrium follows under the conditions guaranteeing the regularity of the associated Jacobian matrix with respect to c_{t+1}^{t+1}, c_{t+1}^t, k_{t+1}, m_t, E_{t+1} of the left hand side of the system of equations above (see Appendix A.1).

At the competitive equilibrium, the two conditions (1) and (2) holding in every period, the agent's budget constraints (16) and (17) guarantee the feasibility of the allocation of resources. Since in any period t, adding up the budget constraints of the young agent, $c_t^t + k_{t+1} + m_t = w_t$, and the contemporaneous old agent, $c_t^{t-1} = r_t k_t$, it holds that

$$c_t^{t-1} + c_t^t + k_{t+1} + m_t = F_K(k_t, 1)k_t + F_L(k_t, 1) = F(k_t, 1)$$

The competitive equilibrium allocation can also be completely characterized by the dynamics of the per capita savings and the dynamics of the per capita investment in environment which result from the agent's utility maximization and the determinations of factor prices.

2.2 Competitive equilibrium steady state

A perfect foresight competitive equilibrium steady state of this overlapping generations economy is a constant sequence $\{k, m\}$ characterized by

$$u'\left(F_L(k,1)-k-m\right)=\frac{\beta(1-b)+\gamma}{\beta+\frac{\gamma}{F_K(k,1)}}v'\left(F_K(k,1)k\right) \tag{21}$$

$$v'\left(F_K(k,1)k\right)=\left(\beta+\frac{\gamma}{F_K(k,1)}\right)\phi'\left(\frac{(\gamma+\beta)m-(\alpha+\beta)F(k,1)+\beta k}{b}\right) \tag{22}$$

the consumptions c_0, c_1 and environmental quality index E at the steady state being determined by

$$c_0=F_L(k,1)-k-m \tag{23}$$

$$c_1=F_K(k,1)k \tag{24}$$

$$E=\frac{(\gamma+\beta)m-(\alpha+\beta)F(k,1)+\beta k}{b} \tag{25}$$

3. Efficient allocation and optimal allocation

In this section, we consider the efficient allocation from the benevolent social planner's point of view. The social planner allocates resources in order to maximize the welfare of both current generation and all future generations. Any allocation selected by her is optimal in the Pareto sense (see Blandchard and Fisher 1989, chapter 3, pp 91 - 104). We will find the efficient allocations and the optimal allocation by solving the dynamic optimization problem below. Assume that the current period is $t=0$, given k_0, E_0, c_0^{-1}, the problem of the social planner is as follows,

$$\underset{\left\{c_t^t,c_{t+1}^t,k_{t+1},m_t,E_{t+1}\right\}_{t=0}^{\infty}}{Max}\sum_{t=0}^{\infty}\frac{u(c_t^t)+v(c_{t+1}^t)+\phi(E_{t+1})}{(1+R)^{t+1}} \tag{26}$$

subject to, $\forall t=0,1,2,...,$

$$F(k_t, 1) = c_t^t + c_t^{t-1} + k_{t+1} + m_t \tag{27}$$

$$E_{t+1} = (1-b)E_t - \alpha F(k_{t+1}, 1) - \beta(c_{t+1}^{t+1} + c_{t+1}^t) + \gamma m_t \tag{28}$$

where $R \geq 0$ is the subjective discount rate of the social planner. The discount rate R is strictly positive when she cares more about the current generation than about the future generations, while R equals to zero when she cares about all generations equally. The first constraint (27) of the problem is the resource constraint of the economy in period t, requiring that the total output is allocated to the consumptions of the young and the old, to savings for the next period's capital stock, and to environmental maintenance. The second constraint (28) is the dynamics of the environmental quality. Solving the problem of the social planner is presented in the Appendix A3.

At the steady state, the FOCs for the social planer's problem can be summarized as follows

$$u'(\bar{c}_0) = \frac{\gamma(1+R) + \beta(1+R)^2}{b+R}\phi'(\bar{E}) \tag{29}$$

$$v'(\bar{c}_1) = \frac{\gamma + \beta(1+R)}{b+R}\phi'(\bar{E}) \tag{30}$$

$$F_K(\bar{k}, 1) = \frac{1+R}{1-(1+R)\alpha/\gamma} \tag{31}$$

The equations of resource constraint and the environmental quality index become

$$F(\bar{k}, 1) = \bar{c}_0 + \bar{c}_1 + \bar{k} + \bar{m} \tag{32}$$

$$\bar{E} = \frac{(\gamma+\beta)\bar{m} - (\alpha+\beta)F(\bar{k}, 1) + \beta\bar{k}}{b} \tag{33}$$

The efficient steady state of this overlapping generations economy can be determined by a constant sequence $\{\bar{c}_0, \bar{c}_1, \bar{k}, \bar{m}, \bar{E}\}$ through solving the system of five equations from (29) to (33).

For the case the social planner cares all generation equally, $R = 0$, the capital ratio at the steady state is the so-called golden rule level of capital per capita. Substituting $R = 0$ into the equations of efficient solution above, the socially optimal allocation is characterized by

$$u'(c_0^*) = \frac{\beta + \gamma}{b}\phi'(E^*) \tag{34}$$

$$v'(c_1^*) = \frac{\beta + \gamma}{b}\phi'(E^*) \tag{35}$$

$$F_K(k^*, 1) = \frac{\gamma}{\gamma - \alpha} \tag{36}$$

$$F(k^*, 1) = c_0^* + c_1^* + k^* + m^* \tag{37}$$

$$E^* = \frac{(\gamma + \beta)m^* - (\alpha + \beta)F(k^*, 1) + \beta k^*}{b} \tag{38}$$

(We assume that $\gamma > \alpha$ which ensures $F_K(k, 1) > 0$, otherwise the evironment would degrade without bound, this seems to be unrealistic)

Diamond (1965) shows that in the standard OLG model without pollution externalities, an economy whose stationary capital per worker exceeds the golden rule level is dynamically inefficient. Gutiérrez (2008) shows that, in an economy, if the pollution externality is large enough then there are always efficient capital ratios that exceed the golden rule capital ratio. She shows the existence of a super golden rule level of capital ratio, beyond the golden rule level, and such that any economy with pollution externalities whose stationary capital ratio exceeds this level is dynamically inefficient. Some notes that should be considered are that: (i) she takes into account pollution externalities from production; (ii) the environment recovers itself overtime at a constant rate; (iii) there is no resource devoted to maintain the environment; (iv) the pollution externality decreases the utility of the agents indirectly by requiring each agent to pay an amount for health cost in the old-age period. In this paper, we consider instead an economy without population growth and pollution externalities coming from both production and consumption; the environment degrade itself over time and there is always an amount devoted to maintain the environment. The quality of environment affects directly the utility of the agents. In contract with Gutiérrez (2008), this paper shows thus that in an economy with pollution externality and without population growth, the golden rule capital ratio is the highest level of capital ratio that is dynamically efficient. This conclusion is accordance with the conclusion of Diamond (1965) for the standard OLG model.

Proposition 1: *In any economy with environmental externalities in which the pollution cleaning technology dominates the pollution marginal effect of production (i.e. $\gamma > \alpha$), the golden rule capital ratio is the highest level that is dynamically efficient.*

Proof:

We know that the efficient capital ratio is implicitly defined to be a function of R by the condition

$$F_K(\bar{k}(R), 1) = \frac{1+R}{1-(1+R)\alpha/\gamma}$$

Since

$$\frac{\partial F_K(\bar{k}(R), 1)}{\partial R} = F_{KK}(\bar{k}(R), 1)\frac{\partial \bar{k}}{\partial R} \tag{39}$$

$$i.e. \quad \frac{\partial \bar{k}}{\partial R} = \frac{1}{F_{KK}(\bar{k}(R), 1)}\frac{\partial F_K(\bar{k}(R), 1)}{\partial R} \tag{40}$$

and $F_{KK}(\bar{k}(R), 1) < 0$ and $\frac{\partial F_K(\bar{k}(R),1)}{\partial R} = \frac{1}{[1-(1+R)\alpha/\gamma]^2} > 0$, hence

$$\frac{\partial \bar{k}(R)}{\partial R} < 0 \tag{41}$$

So, $\bar{k}$ is decreasing in R. Hence, $\bar{k}$ is maximal as $R = 0$, that is exactly the golden rule level of capital. Therefore, $\bar{k}_{max} = k^*$ ■

We have shown in Proposition 1 that any economy with a capital ratio exceeds k^* is dynamically inefficient. It is obvious from (36) that k^* is decreasing in the production pollution parameter α. It is, however, increasing in the environment maintaining technology γ. Hence, economies with more environmental problems coming from production have a larger range of dynamically inefficient allocations. However, the cleaner the environment maintaining technology is, the smaller range of the dynamically inefficient allocations is.

From (34) and (35), the marginal utility of consumption of the young agent must equal that of the consumption of the old agent. The golden rule steady state of this overlapping generations economy is characterized a constant sequence $\{c_0^*,\, c_1^*,\, k^*,\, m^*,\, E^*\}$ solving the system from (34) to (38)

4. Tax Schemes

We have found that a steady state competitive equilibrium is dynamically inefficient when the capital ratio exceeds the golden rule ratio. In this section, we examine how to implement tax and/or transfer policies in order to achieve the optimal allocation in the long run for economies whose competitive equilibrium is dynamically inefficient. Ono (1996) and Gutiérrez (2008) introduced some taxes and transfer schemes to decentralize the first best steady state in the context of pollution externalities. However, their schemes may only hold when the economy already is at the first best steady state. In other words, when the economy is at the first best steady state at some point of time their taxes and transfer policies will help to uphold this state. Nevertheless, one question should be addressed is that *"which policy we can use to help the economy reaching the first best steady state through competitive markets in the transition?"*. In this section we will introduce taxation schemes to help the economy reach the efficient steady state (for the first best steady state, we just set the social planner's discount rate $R = 0$) in the transition and will stay there after reaching the efficient steady state onward. In this paper, such the efficient steady state will be called the best steady state and the corresponding efficient capital ratio is called the best capital ratio. The first best steady state implies the best steady state with $R = 0$. The common strategy of these schemes can be distinguished between two stages. The first stage is the process of transition. In this stage, we choose taxes and transfer such that the capital ratio is always chosen by the agent at the optimal ratio from the social planner's point of view. This stage finishes when the economy converges to a steady state. I will prove that, this steady state completely coincides with the centralized steady state. In the second stage, these schemes will be continuously applied to uphold the steady state. I will present two stages of the first scheme carefully to make the idea easy to follow. Other schemes have similar procedures.

4.1. Taxes on consumptions

Suppose that after finishing the period $t-1$, the economy is reaching the competitive steady state. The social planner needs a tax and transfer scheme to help the economy to go a pathway reaching the best steady state (for given R). This scheme must guarantee that the capital ratio and consumption in period $t+1$ of the agent born in period t always equal to the best steady state capital ratio and the best steady state consumption of the old. Following Ono (1996), consumption taxes are considered. The tax rate of consumption imposed on the young is τ_{0c} which may be different from the tax rate of consumption imposed on the old, τ_{1c}. I also introduce τ_t to be a lump-sum tax levied on the income of the young at date t, and σ_{t+1} to be a lump-sum transfer to the agent when he will be old at date $t+1$. Under this tax system, the problem of an agent born at date t will be

$$\underset{\substack{c_t^t, c_{t+1}^t, k_{t+1}, m_t \geq 0 \\ E_t,\, E_{t+1}^e}}{Max} u(c_t^t) + v(c_{t+1}^t) + \phi(E_{t+1}^e) \tag{42}$$

subject to

$$(1 + \tau_{0c})c_t^t + k_{t+1} + m_t = w_t - \tau_t \tag{43}$$

$$(1 + \tau_{1c})c_{t+1}^t = r_{t+1}k_{t+1} + \sigma_{t+1} \tag{44}$$

$$E_t = (1 - b)E_{t-1} - \alpha F(k_t, 1) - \beta(c_t^t + c_t^{t-1}) + \gamma m_{t-1} \tag{45}$$

$$E_{t+1}^e = (1 - b)E_t - \alpha y_{t+1} - \beta(c_{t+1}^{t+1,e} + c_{t+1}^t) + \gamma m_t \tag{46}$$

Note that in equation (46), $F(k_{t+1}, 1)$ is replaced with y_{t+1} implying that the agent ignores the effect of his savings on the aggregate output. So the agent does not optimizes with respect to k_{t+1} here.

At an equilibrium, the wage rate and capital return will be set at the productivities of labor and capital, respectively. In addition, at a perfect foresight equilibrium the perfect foresight environmental quality is exactly its real value, $E^e_{t+1} = E_{t+1}$. Hence, the first-order condition for this problem can be written as

$$u'(c^t_t) = [\beta(1-b) + \gamma(1+\tau_{0c})]\, \phi'(E_{t+1}) \tag{47}$$

$$v'(c^t_{t+1}) = \left[\beta + \frac{\gamma(1+\tau_{1c})}{F_K(k_{t+1},1)}\right] \phi'(E_{t+1}) \tag{48}$$

By comparing two pairs of equations (47) & (29) and (48) & (30), and considering the best captial ratio given by $F_K(\bar{k},1) = \frac{1+R}{1-(1+R)\alpha/\gamma}$, the consumption tax rates should be set to $\bar{\tau}_{0c} = \frac{\beta+(1-b)(\gamma-\beta b)+\beta R(1+b+R)}{(b+R)\gamma}$ and $\bar{\tau}_{1c} = \frac{(1+R)(\gamma+\beta-\beta b)}{(b+R)(\gamma-\alpha(1+R))} - 1$. These tax rates can be kept unchanged over time. Note that the best steady steady is characterized by $\{\bar{c}_0, \bar{c}_1, \bar{k}, \bar{m}, \bar{E}\}$. At the best steady state, if it can be attained by implementing taxes and transfer scheme, the lump-sum tax and lump-sum transfer are set to constants $\bar{\tau} = F_L(\bar{k},1) - (1+\bar{\tau}_{0c})\bar{c}_0 - \bar{k} - \bar{m}$ and $\bar{\sigma} = (1+\bar{\tau}_{1c})\bar{c}_1 - F_K(\bar{k},1)\bar{k}$, respectively. Obviously, at the best steady state the taxes and transfer scheme guarantees the budget to be balanced, i.e. $\bar{\sigma} = \bar{\tau}_{0c}\bar{c}_0 + \bar{\tau}_{1c}\bar{c}_1 + \bar{\tau}$. We now show that, with tax rates on consumptions above, in any period t there always exists a lump-sum tax, τ_t, imposed on the income of the young in the period t and lump-sum transfer, σ_{t+1}, to the old in the period $t+1$ to ensure that the capital ratio and consumption of the agent, when he old, will be chosen at $\bar{k}$ and $\bar{c}_1$, respectively, by the agent. In effect, for given E_{t-1}, c^{t-1}_t, k_t, m_{t-1}, w_t, $c^{t+1,e}_{t+1}$ and the best capital ratio $\bar{k}$ and best consumption $\bar{c}_1$, let $\{c^t_t, m_t, \tau_t, \sigma_{t+1}, E_t, E_{t+1}\}$ be a solution to the following system of equations

$$(1+\bar{\tau}_{0c})c^t_t + \bar{k} + m_t + \tau_t - F_L(k_t,1) = 0 \tag{49}$$

$$(1+\bar{\tau}_{1c})\bar{c}_1 - F_K(\bar{k},1)\bar{k} - \sigma_{t+1} = 0 \tag{50}$$

$$E_t - (1-b)E_{t-1} + \alpha F(k_t,1) + \beta(c^t_t + c^{t-1}_t + \bar{\tau}_{0c}c^t_t + \tau_t) - \gamma m_{t-1} = 0 \tag{51}$$

$$E_{t+1} - (1-b)E_t + \alpha F(\bar{k}, 1) + \beta(c_{t+1}^{t+1,e} + \bar{c}_1) - \gamma m_t = 0 \tag{52}$$

$$u'(c_t^t) - [\beta(1-b) + \gamma(1+\bar{\tau}_{0c})]\, \phi'(E_{t+1}) = 0 \tag{53}$$

$$v'(\bar{c}_1) - \left[\beta + \frac{\gamma(1+\bar{\tau}_{1c})}{F_K(\bar{k}, 1)}\right] \phi'(E_{t+1}) = 0 \tag{54}$$

Equations (49) and (50) come from the budget constraints of the agent with lump-sum tax and lump-sum transfer. Equations (51) and (52) are evolutions of environment. Note that in the equation (51) the consumption of the old agent now is $\tilde{c}_t^{t-1} = c_t^{t-1} + \bar{\tau}_{0c} c_t^t + \tau_t$ since the old receives a transfer which is exactly equal to what the young agent pays, $\bar{\tau}_{0c} c_t^t + \tau_t$, to keep the government's budget to be balanced. Equations (53) and (54) are derived from the first-order conditions. The existence of a solution is verified by the regularity of the following associated Jacobian matrix J_1 with respect to c_t^t, m_t, τ_t, σ_{t+1}, E_t, E_{t+1}.

$$J_1 = \begin{pmatrix} 1+\bar{\tau}_{0c} & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ \beta(1+\bar{\tau}_{0c}) & 0 & \beta & 0 & 1 & 0 \\ 0 & -\gamma & 0 & 0 & b-1 & 1 \\ u''(c_t^t) & 0 & 0 & 0 & 0 & G_1 \\ 0 & 0 & 0 & 0 & 0 & H_1 \end{pmatrix}$$

with $G_1 = -[\beta(1-b) + \gamma(1+\bar{\tau}_{0c})]\, \phi''(E_{t+1})$, $H_1 = -\left[\beta + \frac{\gamma(1+\bar{\tau}_{1c})}{F_K(\bar{k},1)}\right] \phi''(E_{t+1})$. The existence of a lum-sump tax τ_t and lump-sump transfer σ_{t+1} is stated in the proposition 2.

Proposition 2: *For an overlapping generations economy set up above, in any period of the transition process, there always exists consumption taxes, lump-sum tax and transfer scheme to attain the best capital (saving) ratio $\bar{k}$ and best consumption $\bar{c}_1$ through competitive markets.*

Proof: See Appendix A4.

Note that this scheme of taxes and transfer is merely implementable. In order to implement this scheme precisely, at the beginning of period t, the social planner has to solve the system of equations (49)-(54) given what she knows from the previous period $\{E_{t-1}, c_t^{t-1}, k_t, m_{t-1}\}$, the perfect foresight consumption of the young in the period $t+1$, $c_{t+1}^{t+1,e}$, the wage rate w_t known from labor market, and consumption tax rates $\bar{\tau}_{0c}$, $\bar{\tau}_{1c}$, which she set at the beginning of period t, as well as the best capital ratio $\bar{k}$ and best consumption $\bar{c}_1$ which she is targeting. By solving this system, she will know c_t^t, m_t, τ_t, σ_{t+1}, E_t, E_{t+1} simultaneously. After solving the system she will announce the scheme $\{\bar{\tau}_{0c}, \bar{\tau}_{1c}, \tau_t, \sigma_{t+1}\}$, which has just computed, to the agents. Given this scheme, the agent will behave optimally as the social planner desires.

Proposition 3 will state the taxes and transfer scheme that from the period $t+1$ onward the government's budget will still be always kept balanced and the period $t+1$ is a stepping-stone for economy to achieve the permanent best steady state.

Proposition 3: *After finishing period t (the first stage of taxation), the economy can achieve the best steady state from period $t+1$ onward by implementing the following combination*

$$\bar{\tau}_{0c} = \frac{\beta + (1-b)(\gamma - \beta b) + \beta R(1+b+R)}{(b+R)\gamma} \tag{55}$$

$$\bar{\tau}_{1c} = \frac{(1+R)(\gamma + \beta - \beta b)}{(b+R)(\gamma - \alpha(1+R))} - 1 \tag{56}$$

$$\bar{\tau} = F_L(\bar{k}, 1) - (1 + \bar{\tau}_{0c})\bar{c}_0 - \bar{k} - \bar{m} \tag{57}$$

$$\bar{\sigma} = \bar{\tau}_{0c}\bar{c}_0 + \bar{\tau}_{1c}\bar{c}_1 + \bar{\tau} \tag{58}$$

At such the steady state the government's budget is kept balanced every period.

Proof: See Appendix A5.

4.2. Taxes on consumption and capital income

In the section 4.1, we introduced taxes on consumptions in which the tax rates are different between consumptions of the old and the young. In the reality, however, this tax scheme seems to be difficult to apply because it may violate the equity among generations. In order to avoid the discrimination between the old and the young, an unique rate of consumption tax τ_c should be applied. Beside that, a capital income tax τ_k and a system of lump-sum tax τ_t and lump-sum transfer σ_{t+1} are introduced to show that the best steady state allocation can be achieved. We can also show that the social planner is able to design such the taxes and transfer policy ensuring the government's budget to be balanced. Under this tax system, the problem of an agent in the equilibrium,

$$\underset{\substack{c_t^t, c_{t+1}^t, k_{t+1}, m_t \geq 0 \\ E_t,\, E_{t+1}}}{Max} u(c_t^t) + v(c_{t+1}^t) + \phi(E_{t+1}) \tag{59}$$

subject to

$$F_L(k_t, 1) - \tau_t = (1 + \tau_c)c_t^t + k_{t+1} + m_t \tag{60}$$

$$(1 + \tau_c)c_{t+1}^t = (1 - \tau_k)F_K(k_t, 1)k_{t+1} + \sigma_{t+1} \tag{61}$$

$$E_t = (1 - b)E_{t-1} - \alpha F(k_t, 1) - \beta(c_t^t + c_t^{t-1}) + \gamma m_{t-1} \tag{62}$$

$$E_{t+1}^e = (1 - b)E_t - \alpha y_{t+1} - \beta(c_{t+1}^{t+1,e} + c_{t+1}^t) + \gamma m_t \tag{63}$$

In equation (63), $F(k_{t+1}, 1)$ is replaced with y_{t+1} implying that the agent ignores the effect of his savings on the aggregate output and, therefore, he does not optimizes with respect to k_{t+1} here. At an equilibrium, the wage rate and capital return will be set at the productivities of labor and capital, respectively. In addition, at a perfect foresight equilibrium the perfect foresight environmental quality is exactly its real value, $E_{t+1}^e = E_{t+1}$. Hence, the first-order condition for this problem can be written as

$$u'(c_t^t) = [\beta(1 - b) + \gamma(1 + \tau_c)]\, \phi'(E_{t+1}) \tag{64}$$

$$v'(c_{t+1}^t) = \left[\beta + \frac{\gamma(1 + \tau_c)}{(1 - \tau_k)F_K(k_{t+1}, 1)}\right] \phi'(E_{t+1}) \tag{65}$$

With the same procedures and argument to section 4.1, by comparing two pairs of equations (64) & (29) and (65) & (30), and consider the best captial ratio given by $F_K(\bar{k}, 1) = \frac{1+R}{1-(1+R)\alpha/\gamma}$, the consumption tax rate and capital income tax rate should be set to constants $\bar{\tau}_c = \frac{\beta+(1-b)(\gamma-\beta b)+\beta R(1+b+R)}{(b+R)\gamma}$ and $\bar{\tau}_k = 1 - \frac{(b+R)(\gamma-(1+R)\alpha)(1+\bar{\tau}_c)}{(1+R)(\gamma+\beta-\beta b)}$, respectively. With these tax rates, there always exists a lump-sum tax, τ_t, imposed on the income of the young in the period t and lump-sum transfer, σ_{t+1}, to the old in the period $t+1$ to guarantee the capital ratio and consumption of the agent, when he old, to be chosen at $\bar{k}$ and $\bar{c}_1$, respectively, by the agent. In effect, for given E_{t-1}, c_t^{t-1}, k_t, m_{t-1}, w_t, $c_{t+1}^{t+1,e}$ and the best capital ratio $\bar{k}$ and best consumption $\bar{c}_1$, let $\{c_t^t, m_t, \tau_t, \sigma_{t+1}, E_t, E_{t+1}\}$ be a solution to the following system of equations

$$(1+\bar{\tau}_c)c_t^t + \bar{k} + m_t + \tau_t - F_L(k_t, 1) = 0 \tag{66}$$

$$(1+\bar{\tau}_c)\bar{c}_1 - (1-\bar{\tau}_k)F_K(\bar{k}, 1)\bar{k} - \sigma_{t+1} = 0 \tag{67}$$

$$E_t - (1-b)E_{t-1} + \alpha F(k_t, 1) + \beta(c_t^t + c_t^{t-1} + \bar{\tau}_c c_t^t + \tau_t) - \gamma m_{t-1} = 0 \tag{68}$$

$$E_{t+1} - (1-b)E_t + \alpha F(\bar{k}, 1) + \beta(c_{t+1}^{t+1,e} + \bar{c}_1) - \gamma m_t = 0 \tag{69}$$

$$u'(c_t^t) - [\beta(1-b) + \gamma(1+\bar{\tau}_c)]\,\phi'(E_{t+1}) = 0 \tag{70}$$

$$v'(\bar{c}_1) - \left[\beta + \frac{\gamma(1+\bar{\tau}_c)}{(1-\tau_k)F_K(\bar{k}, 1)}\right]\phi'(E_{t+2}) = 0 \tag{71}$$

The existence of a lump-sump tax and lump-sump transfer scheme can be verified by the regularity of the associated Jacobian matrix J_2 as follows

$$J_2 = \begin{pmatrix} 1+\bar{\tau}_c & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ \beta(1+\bar{\tau}_c) & 0 & 0 & 0 & 1 & 0 \\ 0 & -\gamma & 0 & 0 & b-1 & 1 \\ u''(c_t^t) & 0 & 0 & 0 & 0 & G_2 \\ 0 & 0 & 0 & 0 & 0 & H_2 \end{pmatrix}$$

where $G_2 = -[\beta(1-b) + \gamma(1+\bar{\tau}_c)]\,\phi''(E_{t+1})$, $H_2 = -\left[\beta + \frac{\gamma(1+\bar{\tau}_c)}{(1-\tau_k)F_K(\bar{k},1)}\right]\phi''(E_{t+1}) > 0$. The existence of a lum-sump tax τ_t and lump-sump transfer σ_{t+1} is stated in the proposition 4.

Proposition 4: *For an overlapping generations economy set up above, in any period of the transition process, there always exists consumption taxes, capital income tax, lump-sum tax and transfer scheme to attain the best capital (saving) ratio $\bar{k}$ and best consumption $\bar{c}_1$ through competitive markets.*

Proof: See Appendix A4.

Similar to previous scheme, this scheme is merely implementable. Proposition 5 states that from the period $t+1$ onward the government's budget will still be always kept balanced and the period $t+1$ is a stepping-stone for economy to achieve the permanent best steady state in the period $t+2$ onward.

Proposition 5: *After finishing period t (the first stage of taxation), the economy can achieve the best steady state from period $t+1$ onward by implementing the following combination*

$$\bar{\tau}_c = \frac{\beta + (1-b)(\gamma - \beta b) + \beta R(1+b+R)}{(b+R)\gamma} \tag{72}$$

$$\bar{\tau}_k = 1 - \frac{(b+R)(\gamma - (1+R)\alpha)(1+\bar{\tau}_c)}{(1+R)(\gamma + \beta - \beta b)} \tag{73}$$

$$\bar{\tau} = F_L(\bar{k},1) - (1+\bar{\tau}_c)\bar{c}_0 - \bar{k} - \bar{m} \tag{74}$$

$$\bar{\sigma} = \bar{\tau}_c(\bar{c}_0 + \bar{c}_1) + \bar{\tau}_k F_K(\bar{k},1)\bar{k} + \bar{\tau} \tag{75}$$

At such the steady state the goverment's budget is kept balanced every period.

Proof: See Appendix A5.

4.3 Taxes on consumption and production

We still keep the non-discriminatory tax rate τ_c on consumptions and the system of lump-sum tax τ_t and lump-sum transfer σ_{t+1}. We now introduce a Pigouvian tax on production. In any period, let τ_p be the tax paid by firms per one unit of output produced. We also show that in this scenario the social planner is able to design taxes and transfer policy keeping the government's budget to be balanced and achieving the best allocation through competitive market. The balanced budget implies $\sigma_{t+1} = \tau_c(c_t^t + c_{t+1}^t) + \tau_p F(k_{t+1}, 1) + \tau_t$.

The problem that the firms must solve is

$$\underset{k_t}{Max} \quad (1-\tau_p)F(k_t, 1) - r_t k_t - w_t \tag{76}$$

The return of capital and the return of labor are

$$r_t = (1-\tau_p)F_K(k_t, 1) \tag{77}$$

$$w_t = (1-\tau_p)F_L(k_t, 1) \tag{78}$$

Under this tax scheme, the problem in the equilibrium of an agent born at date t is,

$$\underset{\substack{c_t^t, c_{t+1}^t, k_{t+1}, m_t \geq 0 \\ E_t,\, E_{t+1}}}{Max} u(c_t^t) + v(c_{t+1}^t) + \phi(E_{t+1}) \tag{79}$$

subject to

$$F_L(k_t, 1) - \tau_t = (1+\tau_c)c_t^t + k_{t+1} + m_t \tag{80}$$

$$(1+\tau_c)c_{t+1}^t = (1-\tau_p)F_K(k_{t+1}, 1)k_{t+1} + \sigma_{t+1} \tag{81}$$

$$E_t = (1-b)E_{t-1} - \alpha F(k_t, 1) - \beta(c_t^t + c_t^{t-1}) + \gamma m_{t-1} \tag{82}$$

$$E_{t+1}^e = (1-b)E_t - \alpha y_{t+1} - \beta(c_{t+1}^{t+1,e} + c_{t+1}^t) + \gamma m_t \tag{83}$$

In equation (83), $F(k_{t+1}, 1)$ is replaced with y_{t+1} implying that the agent ignores the effect of his savings on the aggregate output and, therefore, he does not optimizes with respect to k_{t+1} here. At an equilibrium, the wage rate and capital return will be set at the productivities of labor and capital, respectively. In addition, at a perfect foresight equilibrium the perfect foresight environmental quality is exactly its real value, $E_{t+1}^e = E_{t+1}$. With the same procedures and argument to section 4.1, the consumption tax rate and production tax rate should be set to constants $\bar{\tau}_c = \frac{\beta + (1-b)(\gamma - \beta b) + \beta R(1+b+R)}{(b+R)\gamma}$ and

$\bar{\tau}_p = 1 - \frac{(b+R)(\gamma-(1+R)\alpha)(1+\bar{\tau}_c)}{(1+R)(\gamma+\beta-\beta b)}$, respectively. With these tax, there always exists a lump-sum tax imposed on the income of the young to guarantee that the capital ratio and consumption in the old period will be chosen at $\bar{k}$ and $\bar{c}$ by the agent, respectively. In effect, for given E_{t-1}, c_t^{t-1}, k_t, m_{t-1}, w_t, $c_{t+1}^{t+1,e}$ and the best capital ratio $\bar{k}$ and best consumption $\bar{c}_1$, let $\{c_t^t,\ m_t,\ \tau_t,\ \sigma_{t+1},\ E_t,\ E_{t+1}\}$ be a solution to the following system of equations

$$(1+\bar{\tau}_c)c_t^t + \bar{k} + m_t + \tau_t - F_L(k_t,1) = 0 \tag{84}$$

$$(1+\bar{\tau}_c)\bar{c}_1 - (1-\bar{\tau}_p)F_K(\bar{k},1)\bar{k} - \sigma_{t+1} = 0 \tag{85}$$

$$E_t - (1-b)E_{t-1} + \alpha F(k_t,1) + \beta(c_t^t + c_t^{t-1} + \bar{\tau}_c c_t^t + \tau_t) - \gamma m_{t-1} = 0 \tag{86}$$

$$E_{t+1} - (1-b)E_t + \alpha F(\bar{k},1) + \beta(c_{t+1}^{t+1,e} + \bar{c}_1) - \gamma m_t = 0 \tag{87}$$

$$u'(c_t^t) - [\beta(1-b) + \gamma(1+\bar{\tau}_c)]\,\phi'(E_{t+1}) = 0 \tag{88}$$

$$v'(\bar{c}_1) - \left[\beta + \frac{\gamma(1+\bar{\tau}_c)}{(1-\tau_p)F_K(\bar{k},1)}\right]\phi'(E_{t+2}) = 0 \tag{89}$$

The existence of a lump-sum tax and lump-sum transfer scheme can be verified by the regularity of the associated Jacobian matrix J_3 as follows

$$J_3 = \begin{pmatrix} 1+\bar{\tau}_c & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ \beta(1+\bar{\tau}_c) & 0 & 0 & 0 & 1 & 0 \\ 0 & -\gamma & 0 & 0 & b-1 & 1 \\ u''(c_t^t) & 0 & 0 & 0 & 0 & G_3 \\ 0 & 0 & 0 & 0 & 0 & H_3 \end{pmatrix}$$

where $G_3 = -\left[\beta(1-b) + \gamma(1+\bar{\tau}_c)\right]\phi''(E_{t+1})$, $H = -\left[\beta + \frac{\gamma(1+\bar{\tau}_c)}{(1-\tau_p)F_K(\bar{k},1)}\right]\phi''(E_{t+1}) > 0$. The existence of a lump-sum tax τ_t and lump-sum transfer σ_{t+1} is stated in the proposition 6.

Proposition 6: *For an overlapping generations economy set up above, in any period of the transition process, there always exists consumption tax, production tax, lump-sum tax and transfer scheme to attain the best capital (saving) ratio $\bar{k}$ and best consumption $\bar{c}_1$ through competitive markets.*

Proof: See Appendix A4.

This scheme is merely implementable. Proposition 7 states that from the period $t+1$ onward the government's budget will still be always kept balanced and the period $t+1$ is a stepping-stone for economy to achieve the permanent best steady state in the period $t+2$ onward.

Proposition 7: *After finishing period t (the first stage of taxation), the economy can achieve the best steady state from period $t+1$ onward by implementing the following combination*

$$\bar{\tau}_c = \frac{\beta + (1-b)(\gamma - \beta b) + \beta R(1+b+R)}{(b+R)\gamma} \tag{90}$$

$$\bar{\tau}_p = 1 - \frac{(b+R)(\gamma - (1+R)\alpha)(1+\bar{\tau}_c)}{(1+R)(\gamma + \beta - \beta b)} \tag{91}$$

$$\bar{\tau} = (1-\bar{\tau}_p)F_L(\bar{k},1) - \bar{\tau}_c\bar{c}_0 - \bar{k} - \bar{m} \tag{92}$$

$$\bar{\sigma} = \bar{\tau}_c(\bar{c}_0 + \bar{c}_1) + \bar{\tau}_p F(\bar{k},1) + \bar{\tau} \tag{93}$$

At such the steady state the goverment's budget is kept balanced every period.

Proof: See Appendix A5.

4.4 Taxes on consumption, production and labor income

We now modify the tax and transfer policy introduced in section 4.3 by using the labor income tax rate τ_w to replace the lump-sum tax on wage. All other things are kept the same in the section 4.3. The balanced budget condition requires $\sigma_{t+1} = \tau_{wt}w_t + \tau_c(c_t^t + c_t^{t-1}) + \tau_p F(k_t, 1)$.

In equilibrium, the problem of the agent born at date t is

$$\underset{\substack{c_t^t, c_{t+1}^t, k_{t+1}, m_t \geq 0 \\ E_t,\, E_{t+1}}}{Max} u(c_t^t) + v(c_{t+1}^t) + \phi(E_{t+1}) \tag{94}$$

subject to

$$(1 - \tau_{wt})F_L(k_t, 1) = (1 + \tau_c)c_t^t + k_{t+1} + m_t \tag{95}$$

$$(1 + \tau_c)c_{t+1}^t = F_K(k_{t+1}, 1)k_{t+1} + \sigma_{t+1} \tag{96}$$

$$E_t = (1 - b)E_{t-1} - \alpha F(k_t, 1) - \beta(c_t^t + c_t^{t-1}) + \gamma m_{t-1} \tag{97}$$

$$E_{t+1} = (1 - b)E_t - \alpha y_{t+1} - \beta(c_{t+1}^{t+1,e} + c_{t+1}^t) + \gamma m_t \tag{98}$$

In equation (98), $F(k_{t+1}, 1)$ is replaced with y_{t+1} implying that the agent ignores the effect of his savings on the aggregate output and, therefore, he does not optimizes with respect to k_{t+1} here. At an equilibrium, the wage rate and capital return will be set at the productivities of labor and capital, respectively. In addition, at a perfect foresight equilibrium the perfect foresight environmental quality is exactly its real value, $E_{t+1}^e = E_{t+1}$. With the same procedures and argument to section 4.1, the consumption tax rate and production tax rate should be set to constants $\bar{\tau}_c = \frac{\beta+(1-b)(\gamma-\beta b)+\beta R(1+b+R)}{(b+R)\gamma}$ and $\bar{\tau}_p = 1 - \frac{(b+R)(\gamma-(1+R)\alpha)(1+\bar{\tau}_c)}{(1+R)(\gamma+\beta-\beta b)}$, respectively. With these tax, there always exists a labor income tax imposed on the income of the young in period t and a lumpsum transfer to the old in period $t+1$ to guarantee that the capital ratio and consumption in the old period will be chosen at $\bar{k}$ and $\bar{c}$ by the agent, respectively. In effect, for given E_{t-1}, c_t^{t-1}, k_t, m_{t-1}, w_t, $c_{t+1}^{t+1,e}$ and the best capital ratio $\bar{k}$ and best consumption $\bar{c}_1$, let $\{c_t^t,\, m_t,\, \tau_{wt},\, \sigma_{t+1},\, E_t,\, E_{t+1}\}$ be a solution to the following system of equations

$$(1 + \bar{\tau}_c)c_t^t + \bar{k} + m_t - (1 - \tau_{wt})F_L(k_t, 1) = 0 \tag{99}$$

$$(1 + \bar{\tau}_c)\bar{c}_1 - (1 - \bar{\tau}_p)F_K(\bar{k}, 1)\bar{k} - \sigma_{t+1} = 0 \tag{100}$$

$$E_t - (1-b)E_{t-1} + \alpha F(k_t, 1) + \beta(c_t^t + c_t^{t-1} + \bar{\tau}_c c_t^t + \tau_{wt} F_L(k_t, 1)) - \gamma m_{t-1} = 0 \tag{101}$$

$$E_{t+1} - (1-b)E_t + \alpha F(\bar{k},1) + \beta(c_{t+1}^{t+1,e} + \bar{c}_1) - \gamma m_t = 0 \tag{102}$$

$$u'(c_t^t) - [\beta(1-b) + \gamma(1+\bar{\tau}_c)]\,\phi'(E_{t+1}) = 0 \tag{103}$$

$$v'(\bar{c}_1) - \left[\beta + \frac{\gamma(1+\bar{\tau}_c)}{(1-\tau_p)F_K(\bar{k},1)}\right]\phi'(E_{t+2}) = 0 \tag{104}$$

The existence of a labor income tax and lump-sum transfer scheme can be verified by the regularity of the associated Jacobian matrix J_4 as follows

$$J_4 = \begin{pmatrix} 1+\bar{\tau}_c & 1 & F_L(\bar{k},1) & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ \beta(1+\bar{\tau}_c) & 0 & \beta F_L(\bar{k},1) & 0 & 1 & 0 \\ 0 & -\gamma & 0 & 0 & b-1 & 1 \\ u''(c_{t-1}^{t-1}) & 0 & 0 & 0 & 0 & G_4 \\ 0 & 0 & 0 & 0 & 0 & H_4 \end{pmatrix}$$

where $G_4 = -\left[\beta(1-b) + \gamma(1+\bar{\tau}_c)\right]\phi''(E_{t+1})$, $H_4 = -\left[\beta + \frac{\gamma(1+\bar{\tau}_c)}{(1-\tau_p)F_K(\bar{k},1)}\right]\phi''(E_{t+1}) > 0$. The existence of a labor income tax τ_{wt} and lump-sum transfer σ_{t+1} is stated in the proposition 8.

Proposition 8: *For an overlapping generations economy set up above, in any period of the transition process, there always exists consumption tax, production tax, labor income tax and lump-sum transfer scheme to attain the best capital (saving) ratio $\bar{k}$ and best consumption $\bar{c}_1$ through competitive markets.*

Proof: See Appendix A4.

Proposition 9 states that from the period $t+1$ onward the government's budget will still be always kept balanced and the period $t+1$ is a stepping-stone for economy to achieve the permanent best steady state in the period $t+2$ onward.

Proposition 9: *After finishing period t (the first stage of taxation), the economy can achieve the best steady state from period $t+1$ onward by implementing the following combination*

$$\bar{\tau}_c = \frac{\beta + (1-b)(\gamma - \beta b) + \beta R(1+b+R)}{(b+R)\gamma} \tag{105}$$

$$\bar{\tau}_p = 1 - \frac{(b+R)(\gamma - (1+R)\alpha)(1+\bar{\tau}_c)}{(1+R)(\gamma+\beta-\beta b)} \tag{106}$$

$$\bar{\tau}_w = 1 - \frac{(1+\bar{\tau}_c)\bar{c}_0 + \bar{k} + \bar{m}}{(1-\bar{\tau}_p)F_L(\bar{k},1)} \tag{107}$$

$$\bar{\sigma} = \bar{\tau}_w(1-\bar{\tau}_p)F_L(\bar{k},1) + \bar{\tau}_c(\bar{c}_0+\bar{c}_1) + \bar{\tau}_p F_K(\bar{k},1) \tag{108}$$

At such the steady state the goverment's budget is kept balanced every period.

Proof: See Appendix A5.

5 Conclusion

I have presented a general equilibrium overlapping generations model with environmental externalities in which I combined two strains in the literature that the environment is polluted by both production and consumption. For such a model I proved that there exists a competitive equilibrium and then I determined a competitive steady state. This steady state was compared with the best steady state in the social planner's point of view. The pollution externality coming from consumption does not affect the dynamically inefficient range of capital ratio meanwhile the pollution externality coming from production does. The higher the production pollution parameter α, the larger the inefficient range. The environmental maintaining technology γ also plays a role in determining the best steady state capital ratio $\bar{k}$. The cleaner the environment maintaining technology, the smaller range of the dynamically inefficient allocations.

By comparing the competitive steady state and the best steady state, I designed some balanced budget taxes and transfer schemes to decentralize the best steady state. These schemes consist two stages. The first stage is the transition process in which the taxes and transfer schemes help the economy to go a pathway reaching the best steady state. In the second stage, taxes and transfer schemes will uphold the economy always being at the best steady state. I showed that the taxes and transfer schemes are merely implementable if at the beginning of each period the social planner announces the values of taxes and transfer.

This paper simplified many things such as the technology is exogenous, the population growth rate is zero and there is only one production sector, which may be far from the reality. So, there are still many complicated and interesting aspects should be taken into account, which requires to develop the model, in which endogenous technology, endogenous fertility, and the role of human capital accumulation, etc. are still on my research agenda.

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Appendix

A1. Checking for the existence of competitive equilibrium

From the FOCs of the problem of the representative agent and the set of constraints, we have a system defining $(c_{t+1}^{t+1},\ c_{t+1}^{t},\ k_{t+1},\ m_t,\ E_{t+1})$ as a function of $(c_t^t,\ c_t^{t-1},\ k_t,\ m_{t-1},\ E_t)$:

$$c_t^t + k_{t+1} + m_t - F_L(k_t, 1) = 0 \tag{109}$$

$$c_{t+1}^t - F_K(k_{t+1}, 1)k_{t+1} = 0 \tag{110}$$

$$E_{t+1} - (1-b)E_t + \alpha F(k_{t+1}, 1) + \beta(c_{t+1}^{t+1} + c_{t+1}^t) - \gamma m_t = 0 \tag{111}$$

$$u'(c_t^t) - [\beta(1-b) + \gamma]\,\phi'(E_{t+1}) = 0 \tag{112}$$

$$v'(c_{t+1}^t) - \left[\beta + \frac{\gamma}{F_K(k_{t+1}, 1)}\right]\phi'(E_{t+1}) = 0 \tag{113}$$

The non-singularity of the associated Jacobian matrix derived from the system of implicit function guarantees the existence of the competitive equilibrium. The Jacobian matrix of the problem is represented as follows:

$$J = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -C & 0 & 0 \\ \beta & \beta & \alpha F_K(k_{t+1}, 1) & -\gamma & 1 \\ 0 & 0 & 0 & 0 & G \\ 0 & v''(c_{t+1}^t) & D & 0 & H \end{pmatrix}$$

where
$C = F_K(k_{t+1}, 1) + F_{KK}(k_{t+1}, 1)k_{t+1} = \theta^2 A k_{t+1}^{\theta-1} > 0$
$D = \frac{\gamma F_{KK}(k_{t+1},1)}{F_K^2(k_{t+1},1)}\phi'(E_{t+1}) < 0$
$G = -\left[\beta(1-b) + \gamma\right]\phi''(E_{t+1}) > 0$
$H = -\left[\beta + \frac{\gamma}{F_K(k_{t+1},1)}\right]\phi''(E_{t+1}) > 0$

We apply the Laplace's expansion to compute the determinant of the Jacobian matrix. Hence,

$$det(J) = \begin{vmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -C & 0 & 0 \\ \beta & \beta & \alpha F_K(k_{t+1}, 1) & -\gamma & 1 \\ 0 & 0 & 0 & 0 & G \\ 0 & v''(c_{t+1}^t) & D & 0 & H \end{vmatrix}$$

$$= -G \begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & -C & 0 \\ \beta & \beta & \alpha F_K(k_{t+1}, 1) & -\gamma \\ 0 & v''(c_{t+1}^t) & D & 0 \end{vmatrix}$$

$$= -G\beta \left(D + Cv''(c_{t+1}^t) \right) > 0$$

Therefore, the associated Jacobian matrix J is non-singular. This implies there exists a competitive equilibrium, and the dynamics system can be implicitly represented by

$$\begin{pmatrix} c_{t+1}^{t+1} \\ c_{t+1}^t \\ k_{t+1} \\ m_t \\ E_{t+1} \end{pmatrix} = \psi \begin{pmatrix} c_t^t \\ c_t^{t-1} \\ k_t \\ m_{t-1} \\ E_t \end{pmatrix} \tag{114}$$

A2. Checking the SOCs for the maximization problem of the agent

In order the FOCs to be sufficient conditions for them to characterize a (local) maximum to the optimization problem, we have to check the SOCs from the following maximization problem

$$Z = u(c_t^t) + v(c_{t+1}^t) + \phi(E_{t+1}) + \lambda_1(c_t^t + k_{t+1} + m_t - w_t) + \lambda_2(c_{t+1}^t - r_{t+1}k_{t+1})$$

$$+\lambda_3 \left(E_t - (1-b)E_{t-1} + \alpha F(k_t, 1) + \beta(c_t^t + c_t^{t-1}) - \gamma m_{t-1} \right)$$

$$+\lambda_4 \left(E_{t+1} - (1-b)E_t + \alpha F(k_{t+1}, 1) + \beta(c_{t+1}^{t+1,e} + c_{t+1}^t) - \gamma m_t \right)$$

For the problem of the agent, the bordered Hessian will appear as

$$\bar{H} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -r_{t+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & -\gamma & b-1 & 1 \\ 1 & 0 & \beta & 0 & u''(c_t^t) & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \beta & 0 & v''(c_{t+1}^t) & 0 & 0 & 0 & 0 \\ 1 & -r_{t+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \phi''(E_{t+1}) \end{pmatrix}$$

$$(-1)^5 \left|\bar{H}_5\right| = -\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \beta & 0 & u''(c_t^t) \end{vmatrix} = 0$$

$$(-1)^6 \left|\bar{H}_6\right| = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \\ 1 & 0 & \beta & 0 & u''(c_t^t) & 0 \\ 0 & 1 & 0 & \beta & 0 & v"(c_{t+1}^t) \end{vmatrix} = 0$$

$$(-1)^7 \left|\bar{H}_7\right| = -\begin{vmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -r_{t+1} \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 \\ 1 & 0 & \beta & 0 & u''(c_t^t) & 0 & 0 \\ 0 & 1 & 0 & \beta & 0 & v''(c_{t+1}^t) & 0 \\ 1 & -r_{t+1}^e & 0 & 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

$$(-1)^8 \left|\bar{H}_8\right| = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -r_{t+1} & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & -\gamma \\ 1 & 0 & \beta & 0 & u''(c_t^t) & 0 & 0 & 0 \\ 0 & 1 & 0 & \beta & 0 & v''(c_{t+1}^t) & 0 & 0 \\ 1 & -r_{t+1}^e & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\gamma & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$= \left[det \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -r_{t+1} & 0 \\ \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & -\gamma \end{pmatrix} \right]^2 = \left(\beta^2 r_{t+1} + \gamma\beta\right)^2 > 0$$

$$(-1)^9 \left|\bar{H}_9\right| = -\begin{vmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -r_{t+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & -\gamma & b-1 \\ 1 & 0 & \beta & 0 & u''(c_t^t) & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \beta & 0 & v''(c_{t+1}^t) & 0 & 0 & 0 \\ 1 & -r_{t+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b-1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$= -\left(\beta(1-b)+\gamma\right)^2 r_{t+1}^2 v''(c_{t+1}^t) - \left(\beta r_{t+1}+\gamma\right)^2 u''(c_t^t) > 0$$

$$(-1)^{10} \left|\bar{H}_{10}\right| = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -r_{t+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & -\gamma & b-1 & 1 \\ 1 & 0 & \beta & 0 & u''(c_t^t) & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \beta & 0 & v''(c_{t+1}^t) & 0 & 0 & 0 & 0 \\ 1 & -r_{t+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \phi''(E_{t+1}) \end{vmatrix}$$

$$= r_{t+1}^2 v''(c_{t+1}^t) u''(c_t^t) + \phi''(E_{t+1}) \left|\bar{H}_9\right| > 0$$

The properties of SOEs guarantees the solution of the agent is a maximal.

A3. Solving the problem of the social planner

The Lagrange function for this problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \frac{u(c_t^t) + u(c_{t+1}^t) + \phi(E_{t+1})}{(1+R)^{t+1}} + \sum_{t=0}^{\infty} \frac{\mu_t \left[F(k_t,1) - c_t^t - c_t^{t-1} - k_{t+1} - m_t\right]}{(1+R)^{t+1}}$$

$$+ \sum_{t=0}^{\infty} \frac{\eta_t \left[E_{t+1} - (1-b)E_t + \alpha F(k_{t+1},1) + \beta(c_{t+1}^{t+1} + c_{t+1}^t) - \gamma m_t\right]}{(1+R)^{t+1}} \tag{115}$$

The FOCs of the maximazation problem are

$$\frac{\partial \mathcal{L}}{\partial c_t^t} = \frac{u'(c_t^t)}{(1+R)^{t+1}} - \frac{\mu_t}{(1+R)^{t+1}} + \frac{\beta\eta_{t-1}}{(1+R)^t} = 0 \tag{116}$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}^t} = \frac{v'(c_{t+1}^t)}{(1+R)^{t+1}} - \frac{\mu_{t+1}}{(1+R)^{t+2}} + \frac{\beta\eta_t}{(1+R)^{t+1}} = 0 \tag{117}$$

$$\frac{\partial \mathcal{L}}{\partial E_{t+1}} = \frac{\phi'(E_{t+1})}{(1+R)^{t+1}} + \frac{\eta_t}{(1+R)^{t+1}} - \frac{(1-b)\eta_{t+1}}{(1+R)^{t+2}} = 0 \tag{118}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = \frac{\mu_{t+1}F_K(k_{t+1},1)}{(1+R)^{t+2}} + \frac{\eta_t\alpha F_K(k_{t+1},1)}{(1+R)^{t+1}} - \frac{\mu_t}{(1+R)^{t+1}} = 0 \tag{119}$$

$$\frac{\partial \mathcal{L}}{\partial m_t} = -\frac{\mu_t}{(1+R)^{t+1}} - \frac{\eta_t\gamma}{(1+R)^{t+1}} = 0 \tag{120}$$

At the steady state,

$$u'(\bar{c}_0) = \mu - \beta\eta(1+R)$$

$$v'(\bar{c}_1) = \frac{\mu}{1+R} - \beta\eta$$

$$\phi'(\bar{E}) = -\eta + \frac{(1-b)\eta}{1+R}$$

$$F_K(\bar{k},1) = \frac{\mu(1+R)}{\mu + \alpha\eta(1+R)}$$

$$\mu = -\eta\gamma$$

Therefore,

$$u'(\bar{c}_0) = \frac{\gamma(1+R) + \beta(1+R)^2}{b+R}\phi'(\bar{E})$$

$$v'(\bar{c}_1) = \frac{\gamma + \beta(1+R)}{b+R}\phi'(\bar{E})$$

$$F_K(\bar{k},1) = \frac{1+R}{1-(1+R)\alpha/\gamma}$$

A4. Proof of proposition 2, proposition 4, proposition 6, proposition 8

It is clearly to find that, the associated Jacobian matrixes J_1, J_2 and J_3 are similar. The matrix J_4 is nearly the same to others. However, by following the same steps presented below to verified the regularity of J_1, then the regularity of J_4 will be also verified easily. So, it suffices to prove proposition 2 only. The remaining others could be proved completely in the same way.

The consumption tax rates are $\bar{\tau}_{0c} = \frac{\beta+(1-b)(\gamma-\beta b)+\beta R(1+b+R)}{(b+R)\gamma}$ and $\bar{\tau}_{1c} = \frac{(1+R)(\gamma+\beta-\beta b)}{(b+R)(\gamma-\alpha(1+R))} - 1$. Now, I show the existence of the lump-sum tax and and hence lump-sum is defined by showing the regularity of the associated Jacobian matrix J_1. In effect,

$$det(J_1) = \begin{vmatrix} 1+\bar{\tau}_{0c} & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ \beta(1+\bar{\tau}_{0c}) & 0 & \beta & 0 & 1 & 0 \\ 0 & -\gamma & 0 & 0 & b-1 & 1 \\ u''(c_t^t) & 0 & 0 & 0 & 0 & G_1 \\ 0 & 0 & 0 & 0 & 0 & H_1 \end{vmatrix}$$

$$= H_1 \begin{vmatrix} 1+\bar{\tau}_{0c} & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \beta(1+\bar{\tau}_{0c}) & 0 & \beta & 0 & 1 \\ 0 & -\gamma & 0 & 0 & b-1 \\ u''(c_t^t) & 0 & 0 & 0 & 0 \end{vmatrix} = H_1 u''(c_t^t) \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \beta & 0 & 1 \\ -\gamma & 0 & 0 & b-1 \end{vmatrix}$$

$$= H_1 u''(c_t^t) \begin{vmatrix} 1 & 1 & 0 \\ 0 & \beta & 1 \\ -\gamma & 0 & b-1 \end{vmatrix} = -H_1 u''(c_t^t)\left[\beta(1-b)+\gamma\right] > 0$$

Hence, J_1 is regular $\blacksquare$.

A5. Proofs for proposition 3, proposition 5, proposition 7, proposition 9

The proofs for proposition 3, proposition 5, proposition 7, proposition 9 have similar ideas and procedures. It suffices to prove proposition 3 only.

Proof for proposition 3

I show that from the period $t+1$ onward the government's budget will still be always kept balanced and the period $t+1$ is a stepping-stone for economy to achieve permanently the best steady state in the period $t+2$ onward. In effect, equations (54) and (30) imply $E_{t+1} = \bar{E}$, and then (53) and (29) imply $c_t^t = \bar{c}_0$. The system of equations to determine the lump-sum tax and lump-sum transfer for the agent born in period $t+1$ becomes

$$(1+\bar{\tau}_{0c})c_{t+1}^{t+1} + \bar{k} + m_{t+1} + \tau_{t+1} - F_L(\bar{k}, 1) = 0 \tag{121}$$

$$(1+\bar{\tau}_{1c})\bar{c}_1 - F_K(\bar{k}, 1)\bar{k} - \sigma_{t+2} = 0 \tag{122}$$

$$E_{t+1}(= \bar{E}) = (1-b)E_t - \alpha F(\bar{k}, 1) - \beta(c_{t+1}^{t+1} + \bar{c}_1) + \gamma m_t \tag{123}$$

$$E_{t+2} - (1-b)E_{t+1} + \alpha F(\bar{k}, 1) + \beta(c_{t+2}^{t+2,e} + \bar{c}_1) - \gamma m_{t+1} = 0 \tag{124}$$

$$u'(c_{t+1}^{t+1}) - [\beta(1-b) + \gamma(1+\bar{\tau}_{0c})]\, \phi'(E_{t+2}) = 0 \tag{125}$$

$$v'(\bar{c}_1) - \left[\beta + \frac{\gamma(1+\bar{\tau}_{1c})}{F_K(\bar{k}, 1)}\right] \phi'(E_{t+2}) = 0 \tag{126}$$

The existence of $\{c_{t+1}^{t+1},\, m_{t+1},\, \tau_{t+1},\, \sigma_{t+2},\, E_{t+1},\, E_{t+2}\}$ is verified easily by the regularity of the associated Jacobian matrix J_1'.

$$J_1' = \begin{pmatrix} 1+\bar{\tau}_{0c} & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ \beta & 0 & 0 & 0 & 1 & 0 \\ 0 & -\gamma & 0 & 0 & b-1 & 1 \\ u''(c_{t+1}^{t+1}) & 0 & 0 & 0 & 0 & G_1' \\ 0 & 0 & 0 & 0 & 0 & H_1' \end{pmatrix}$$

Again, equation (30) and (126) imply $E_{t+2} = \bar{E}$, then at a perfect foresight equilibrium (29) and (125) imply $c_{t+1}^{t+1} = c_{t+1}^{t+1,e} = \bar{c}_0$. It is also similar to prove that, at a perfect foresight equilibrium, this condition $c_{t+2}^{t+2} = c_{t+2}^{t+2,e} = \bar{c}_0$ holds. Hence, equation (124) implies $m_{t+1} = \bar{m}$. Substituting $c_{t+1}^{t+1} = \bar{c}_0$, $m_{t+1} = \bar{m}$ into (121) we have $\tau_{t+1} = \bar{\tau}$. And $\sigma_{t+1} = \bar{\sigma} = \bar{\tau}_{0c}\bar{c}_0 + \bar{\tau}_{1c}\bar{c}_1 + \bar{\tau} = \bar{\tau}_{0c}c_{t+1}^{t+1} + \bar{\tau}_{1c}\bar{c}_1 + \tau_{t+1}$ implies that the government's budget in period $t+1$ is balanced. Moreover, the economy achieves the best steady state in period $t+1$. From period $t+2$ onward the balanced taxes and transfer scheme $\{\bar{\tau}_{0c},\, \bar{\tau}_{1c},\, \bar{\tau},\, \bar{\sigma}\}$ will be implemented to uphold the best steady state permanently. ■