Equilibrium in International Asset Market

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1 The Model

Let us consider a two-period economy with $L+1$ countries and K assets. We suppose there exists one consumption good which may be traded between the $L+1$ countries. In each country there is only one consumer. In period 0, agent i, $(i = 0, \ldots, L)$ purchases assets and consumes in period 1. There are S states of nature in period 1. If state s occurs, in period 1, the consumer in country i will consume c^i_s:

$$c^i_s = \omega^i_s + \sum_{k=1}^{K} R^i_k(s)\theta^i_k$$

where θ^i is the portfolio she purchased in period 0, ω^i_s is the initial endowment of consumption good, $R^i_k(s) \geq 0$ is the return of asset k in country i. The initial endowment ω^i_s and the return $R^i_k(s)$ are valued in currency of country i.
We make the following assumptions:
A1: For any i, any s, $\sum_{k=1}^{K} R^i_k(s) > 0$
A2: For any i, any k, $\sum_{s=1}^{S} R^i_k(s) > 0$
These assumptions are not very stringent. If **A1** is not satisfied for some i, some s, in this case, country i will not make any exchange on the asset market in state s. If **A2** is not satisfied for some i, some k, country i will never purchase asset k.
P: For every state s, every country i, $\pi^i_s > 0$, where π^i_s is the belief probability of agent i that state s occurs in period 1.

For any state s, we take the consumption price of country 0 as numeraire. Hence, if τ^i_s is the exchange rate in state s between countries i and 0, the consumption of agent i valued in currency 0 is

$$\tau^i_s c^i_s = \tau^i_s\omega^i_s + \sum_{k=1}^{K} \tau^i_s R^i_k(s)\theta^i_k$$

2 The consumption model

We first consider the two-period consumption model. Let $(\pi_s^i \neq 0)$ in the S-unit simplex be the belief of agent i. If q is the asset price, agent i will solve:

$$(\mathcal{P}) \qquad \max \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k^i)$$
$$\forall s,\ \omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k^i \geq 0$$
$$\sum_{k=1}^{K} q_k \theta_k^i \leq 0.$$

We suppose that for any i, agent i has no initial endowment for the assets. We also asume

U1: The utility function u^i is concave, strictly increasing, differentiable in $\mathbb{R}_{++}$ for any i.

Definition 1

An equilibrium is a list $[(\theta^{*i}, (c_s^{*i}; \tau_s^{*i})_{s=1,\dots,S})_{i=0,\dots,L}, q^* \neq 0]$ such that
(1) $\forall i,\ \theta^{*i}$ will solve problem $(\mathcal{P})$ given q^*
(2) $\sum_{i=0}^{L} \theta^{*i} = 0$
(3) $\sum_{i=0}^{L} \tau_s^{*i} c_s^{*i} = \sum_{i=0}^{L} \tau_s^{*i} \omega_s^i$.
Relation (2) is the market clearing on the asset market while relation (3) is the balance on the consumption good market valued in currency 0.

Remark

By definition, $\tau_s^{*0} = 1$ for any s.
It is obvious that the consumption set X^i is

$$X^i = \left\{ \theta \in \mathbb{R}^K : \text{for any } s,\ \omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k \geq 0 \right\}$$

Definition 2

w is a useful assets purchase for agent i if for any $\lambda \geq 0$, for any $\theta \in X^i$, one has:

$$(a) \qquad \omega_s^i + \sum_{k=1}^{K} R_k^i(s)(\theta_k + \lambda w_k) \geq 0$$
$$(b) \qquad \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)(\theta_k + \lambda w_k) \geq \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k)$$

Let W^i denote the set of useful vectors for agent i.

Proposition 1 *We have*

$$W^i = \left\{ w \in \mathbb{R}^K : \sum_{k=1}^{K} R_k^i(s) w_k \geq 0, \ \forall s \right\}$$

Proof: Consider (a) in the previous definition. Divide the LHS by λ and let it go to infinity. We obtain $\sum_{k=1}^{K} R_k^i(s) w_k \geq 0$.
Conversely, assume $\forall s, \sum_{k=1}^{K} R_k^i(s) w_k \geq 0$. Then obviously, for any $\theta \in X^i$, any $\lambda \geq 0$, one has (a). From the increasingness of u^i, we have

$$\sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)(\theta_k + \lambda w_k) \geq \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k)$$

We obtain (b). ∎

Definition 3
A vector q is a no-arbitrage price for agent i if $q \cdot w > 0$, for all $w \in W^i$.
Let S^i denote the cone of no-arbitrage prices for agent i. Then, obviously, $S^i = -\ \mathrm{int}(W^i)^0$.
In finance, there is another concept of no-arbitrage. We call it NA1. A vector q is a $NA1$ price, or more simply $NA1$, if for any country i, for any portfolio θ which satisfies $R_k^i(s) \cdot \theta \geq 0, \ \forall s$, and $R_k^i(s') \cdot \theta > 0$ for some s', then we have $q \cdot \theta > 0$.
We introduce an assumption on the exchange rates:
$(\mathcal{E})$ There exist $[(\tau_s^{*i} > 0); i = 1, \ldots, L; s = 1, \ldots, S]$, $\tau_s^{*0} = 1; s = 1, \ldots, S$ such that

$$\forall i \neq 0, \ \forall s, \forall k, \ \tau_s^{*i} R_k^i(s) = R_k^0(s).$$

Proposition 2 *Under* $(\mathcal{E})$*, a vector* q *is NA1 iff,* $\forall s, \ R_k^0(s) \cdot \theta \geq 0$ *and* $R_k^0(s') \cdot \theta > 0$ *for some* s'*, then* $q \cdot w > 0$*.*

Proof: Obvious. ∎

Proposition 3 *Assume **A1, A2, P, U1** and* $\omega_s^i > 0, \forall s, \ \forall i$*. Assume moreover the no-arbitrage condition*

$$(\mathcal{NA}) \ \cap_{i=0}^{L} S^i \neq \emptyset$$

Then there exist $[(\theta^{*i})_{i=0,\ldots,L}; q^* >> 0]$ *such that*
(a) $\forall i, \ \theta^{*i}$ *solves problem* $(\mathcal{P})$
(b) $\sum_{i=0}^{L} \theta^{*i} = 0$

Proof: The proof may be found in several papers, e.g., Werner [11], Page and Wooders [8], Dana, Le Van, Magnien [5]. The strict positivity of q^* comes from the strict increasingness of the u^i and assumptions **A1, A2**. ■

We now introduce an assumption which ensures the non-emptiness of S^i.
A3: For any i, there exists no non-null $(\theta_1, \ldots, \theta_K)$ which satisfies

$$\forall s, \ \sum_{k=1}^{K} R_k^i(s)\theta_k = 0$$

This assumption means that, for any country i, the K assets are not redundant. With this assumption, for any i, W^i contains no line, and S^i is non-empty.

Proposition 4 *Assume A3. Then q is NA1 iff it is a no-arbitrage price.*

Proof: Let q be no-arbitrage. Given i, let w satisfy $R_k^i(s) \cdot w \geq 0, \ \forall s$ and $R_k^i(s') \cdot w > 0$ for some s'. In this case $w \in W^i \setminus \{0\}$. Hence $q \cdot w > 0$. That means q is NA1.
Conversely, let q be NA1. Given i, let $w \in W^i \setminus \{0\}$. then we have $R_k^i(s) \cdot w \geq 0, \ \forall s$ and $R_k^i(s') \cdot w > 0$ for some s'. If not, $R_k^i(s) \cdot w = 0, \ \forall s$ and from A3, $w = 0$: a contradiction. Since q is NA1, we have $q \cdot w > 0$, i.e. q is no-arbitrage. ■

Proposition 5 *(a) If q^* is an equilibrium price then it is NA1.*
(b) Assume A3. If q^* is an equilibrium price then it is both NA1 and no-arbitrage.

Proof: (a) Given i, let ψ satisfy $R_k^i(s) \cdot \psi \geq 0, \ \forall s$ and $R_k^i(s') \cdot \psi > 0$ for some s'. Let θ^{*i} denote the associated equilibrium portfolio. Since u^i is strictly increasing, and $\pi_s^i > 0, \ \forall s$, we have

$$\sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)(\theta_k^{*i} + \psi_k)) > \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k^{*i})$$

That implies $q \cdot \psi > 0$.
(b) The result follows from (a) and Proposition 4. ■

Proposition 6 *Assume A1, A2, A3, P, U1 and $\mathcal{E}$. Assume that for any i, $\omega_s^i > 0, \forall s$. Then there exist $[(\theta^{*i})_{i=0,\ldots,L}; q^* \neq 0]$ such that $[(\theta^{*i})_{i=0,\ldots,L}; (\tau_s^{*i})_{i,s}; q^* \neq 0]$ is an equilibrium.*

Proof: Under $(\mathcal{E})$, the set W^i

$$\begin{aligned} W^i &= \left\{ w \in \mathbb{R}^K : \sum_{k=1}^{K} R_k^i(s) w_k \geq 0, \ \forall s \right\} \\ &= \left\{ w \in \mathbb{R}^K : \frac{1}{\tau_s^{*i}} \sum_{k=1}^{K} R_k^0(s) w_k \geq 0, \ \forall s \right\} \\ &= \left\{ w \in \mathbb{R}^K : \sum_{k=1}^{K} R_k^0(s) w_k \geq 0, \ \forall s \right\} \end{aligned}$$

is independent of i and hence S^i is the same for all i. We will show that S^0 is non-empty. Indeed, let $w \in W^0 \setminus \{0\}$. Then there exists s' such that $\sum_{k=1,\ldots,K} R_k^0(s') w_k > 0$. If not, we have: $\forall s, \ \sum_{k=1,\ldots,K} R_k^0(s) w_k = 0$. From **A3**, $w = 0$ which is a contradiction. Now, let $q \in \mathbb{R}^K$ be defined by $\forall k, \ q_k = \sum_{s=1,\ldots,S} R_k^0(s)$. Then $q \cdot w > 0$ for any $w \in W^0 \setminus \{0\}$. That means $q \in S^0$. The No-Arbitrage condition $(\mathcal{NA})$ is therefore satisfied.

From Proposition 3, there exist $[(\theta^{*i})_{i=0,\ldots,L}; q^* \neq 0]$ such that

$$\begin{aligned} &(a) \quad \forall i, \ \theta^{*i} \text{ solves problem } (\mathcal{P}) \\ &(b) \quad \sum_{i=0}^{L} \theta^{*i} = 0. \end{aligned}$$

Condition $(\mathcal{E})$ implies

$$(c) \ \sum_{i=0}^{L} \tau_s^{*i} c_s^{*i} = \sum_{i=0}^{L} \tau_s^{*i} \omega_s^i$$

where

$$c_s^{*i} = \omega_s^i + \sum_{k=1}^{K} R_k^i(s) \theta_k^{*i}.$$

We end the proof. ■

3 The wealth model

We drop the constraints

$$\forall s, \ \omega_s^i + \sum_{k=1}^{K} R_k^i(s) \theta_k^i \geq 0$$

We replace **U1** by

U1bis: For every i, the utility function is is concave, strictly increasing, differentiable in $\mathbb{R}$

Let $a^i = u^{i\prime}(+\infty)$, $b^i = u^{i\prime}(-\infty)$, $i = 0, \ldots, L$. Let q be the asset price. Country i will solve:

$$(\mathcal{Q}) \qquad \max \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k^i)$$
$$\sum_{k=1}^{K} q_k \theta_k^i \leq 0.$$

We suppose, as in the two-period consumption model, that for any i, agent i has no initial endowment for the assets.

Definition 4

An equilibrium is a list $[(\theta^{*i}, (c_s^{*i}; \tau_s^{*i})_{s=1,\ldots,S})_{i=0,\ldots,L}, q^* \neq 0]$ such that

(1) $\forall i$, θ^{*i} will solve problem $(\mathcal{Q})$ given q^*

(2) $\sum_{i=0}^{L} \theta^{*i} = 0$

(3) $\sum_{i=0}^{L} \tau_s^{*i} c_s^{*i} = \sum_{i=0}^{L} \tau_s^{*i} \omega_s^i$.

Relation (2) is the market clearing on the asset market while relation (3) is the balance on the consumption good market valued in currency 0.

Remark By definition, as before, $\tau_s^{*0} = 1$ for any s.

It is obvious that the consumption set X^i is now $\mathbb{R}^K$.

In order to prove existence of equilibrium, we will introduce and characterize the useful vectors and the no-arbitrage prices.

Definition 5

w is a useful assets purchase for agent i if for any $\lambda \geq 0$, for any $\theta \in X^i$, one has:

$$\sum_{s=1}^{S} \pi_s^i u^i \left(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)(\theta_k + \lambda w_k) \right) \geq \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k)$$

From Rockafellar [10], w is useful for agent i, if, and only if, there exists $\theta \in X^i$ such that

$$\sum_{s=1}^{S} \pi_s^i u^i \left(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)(\theta_k + \lambda w_k) \right) \geq \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k), \ \forall \lambda \geq 0.$$

Let W^i denote the set of useful vectors for agent i. We will characterize it.

Proposition 7 *A vector w is useful for i iff:*

$$\forall \theta \in \mathbb{R}^K, \ \sum_{k'=1}^{K} w_{k'} \sum_{s=1}^{S} \pi_s^i u^{i\prime}(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k) R_{k'}^i(s) \geq 0 \tag{1}$$

Proof: It is very similar to those given in Dana and Le Van [3], [4] by using the concavity and the differentiability of the u^i. ■

We can have another characterization of W^i. The proof of the following proposition is adapted from Dana and Le Van [4].

Proposition 8 *Let $w \in X^i$ and let $\zeta_s = \sum_k R^i_k(s)w_k,\ \forall s,\ S^+ = \{s : \zeta_s > 0\}$, $S^- = \{s : \zeta_s < 0\}$. The vector w is useful is for i iff*

$$a^i \sum_{s \in S^+} \pi^i_s \zeta_s + b^i \sum_{s \in S^-} \pi^i_s \zeta_s \geq 0 \tag{2}$$

Proof: From Proposition 7, w is useful iff for any $\theta \in \mathbb{R}^K$, we have

$$\sum_{s=1}^{S} \pi^i_s u^i \left(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)(\theta_k + \lambda w_k) \right) \geq \sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)\theta_k),\ \forall \lambda \geq 0.$$

Take $\theta = 0$. We then have

$$\sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \lambda \zeta_s) \geq \sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s),\ \forall \lambda \geq 0.$$

Thus, ζ is useful for the function $(c_s)_s \to \sum \pi^i_s u^i(c_s)$. We then have for any $(c_s)_s$

$$0 \geq \sum_{s=1}^{S} \pi^i_s u^i(c_s) - \sum_{s=1}^{S} \pi^i_s u^i(c_s + \zeta_s) \geq - \sum_{s=1}^{S} \pi^i_s u^{i\prime}(c_s)\zeta_s.$$

This implies $\sum_{s=1}^{S} \pi^i_s u^{i\prime}(c_s)\zeta_s \geq 0$. For any $s \in S^+$ let c_s go to $+\infty$, and for $s \in S^-$, let c_s go $-\infty$. We then obtain (2).

The converse is obvious since $u^{i\prime}$ is non-increasing. ∎

Corollary 1 *If $a^i = 0$ or $b^i = +\infty$ then $W^i = \left\{ w \in \mathbb{R}^K : \sum_{k=1}^{K} R^i_k(s)w_k \geq 0,\ \forall s \right\}$*

Proof: It is obvious. ∎

As before, we define no-arbitrage prices.

Definition 6

A vector q is a no-arbitrage price for agent i if $q \cdot w > 0$, for all $w \in W^i$. A vector q is called no-arbitrage price if it is for every i.

Let S^i denote the cone of no-arbitrage prices for agent i. Then, obviously, $S^i = -\ \mathrm{int}(W^i)^0$.

In finance, there is another concept of no-arbitrage. We call it NA1. A vector q is a $NA1$ price, or more simply $NA1$, if for any country i, for any portfolio θ which satisfies $R^i_k(s) \cdot \theta \geq 0,\ \forall s$, and $R^i_k(s') \cdot \theta > 0$ for some s', then we have $q \cdot \theta > 0$.

Proposition 9 *(a) If q is no-arbitrage, then it is NA1. If q^* is an equilibrium price, then it is NA1.*
(b) Assume A3. If u^i is strictly concave then

$$\sum_s \pi_s^i u^i(\omega_s^i + \sum_k R_k^i(s)(\theta_k + w_k)) > \sum_s \pi_s^i u^i(\omega_s^i + \sum_k R_k^i(s)\theta_k)$$

for any θ, any $w \in W^i \setminus \{0\}$. And any equilibrium price is no-arbitrage.

Proof: (a)Let q be no-arbitrage. Given i, let w satisfy $R_k^i(s) \cdot w \geq 0$, $\forall s$ and $R_k^i(s') \cdot w > 0$ for some s'. In this case $w \in W^i \setminus \{0\}$. Hence $q \cdot w > 0$. That means q is NA1.
Given i, let ψ satisfy $R_k^i(s) \cdot \psi \geq 0$, $\forall s$ and $R_k^i(s') \cdot \psi > 0$ for some s'. Let θ^{*i} denote the associated equilibrium portfolio. Since u^i is strictly increasing, and $\pi_s^i > 0$, $\forall s$, we have

$$\sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)(\theta_k^{*i} + \psi_k)) > \sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)\theta_k^{*i})$$

That implies $q \cdot \psi > 0$.
(b) Let $w \in W^i \setminus \{0\}$. Then from A3, $\sum_s R_k^i(s) w_k \neq 0$. If

$$\sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)(\theta_k^i + w_k)) = \sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)\theta_k^i)$$

then

$$\sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)(\theta_k^i + \frac{1}{2}w_k)) > \sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)(\theta_k^i + w_k))$$

which is a contradiction since

$$\sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)(\theta_k^i + w_k)) \geq \sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)(\theta_k^i + \frac{1}{2}w_k))$$

Let $[(\theta^{*i}), q^*]$ be an equilibrium. Then for any $w \in W^i \setminus \{0\}$ we have

$$\sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)(\theta_k^{*i} + w_k)) > \sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)\theta_k^{*i})$$

This implies $q^* \cdot w > 0$. ■

We have the first result for existence of equilibrium.

Proposition 10 *Assume A1, A2, A3, P, U1bis, condition ($\mathcal{E}$), and for any i, either $a^i = 0$ or $b^i = +\infty$. Then there exists an equilibrium.*

Proof: In this case, for any i, $W^i = \{w \in \mathbb{R}^K : \sum_s R^i_k(s)w_k \geq 0,\ \forall s\}$. The proof is the same as for Proposition 6. ■

More generally,

Proposition 11 *Assume A1, A2, A3, P, U1bis, condition* $(\mathcal{E})$*, and for any* i*,* $a^i < u^{i\prime}(\omega^i_s + \sum_{k=1}^K R^i_k(s)\theta_k) < b^i$*,* $\forall\theta$*. Then there exists an equilibrium iff there exists a no-arbitrage price, i.e. there exist* $[(\theta^i, \lambda^i > 0)_{i=0,\dots,L}]$ *such that*

$$\forall i,\ \forall j,\ \forall k',\ \lambda^i \sum_s \pi^i_s u^{i\prime}(\omega^i_s + \sum_k R^i_k(s)\theta^i_k)R^i_{k'} = \lambda^j \sum_s \pi^j_s u^{j\prime}(\omega^j_s + \sum_k R^j_k(s)\theta^j_k)R^j_{k'}$$

Proof: (1) Assume there exist $[(\theta^i, \lambda^i > 0)_{i=0,\dots,L}]$ such that

$$\forall i,\ \forall j,\ \forall k',\ \lambda^i \sum_s \pi^i_s u^{i\prime}(\omega^i_s + \sum_k R^i_k(s)\theta^i_k)R^i_{k'} = \lambda^j \sum_s \pi^j_s u^{j\prime}(\omega^j_s + \sum_k R^j_k(s)\theta^j_k)R^j_{k'}$$

Let

$$q_{k'} = \lambda^i \sum_s \pi^i_s u^{i\prime}(\omega^i_s + \sum_k R^i_k(s)\theta^i_k)R^i_{k'},\ \forall k'$$

We will show that q is no-arbitrage. Indeed, let $w \in W^i \setminus \{0\}$. Let $\zeta_s = \sum_k R^i_k(s)w_k$, $\forall s$. We will show

$$q \cdot w = \lambda^i \sum_s \pi^i_s u^{i\prime}(\omega^i_s + \sum_k R^i_k(s)\theta^i_k)\zeta_s > 0$$

From A3, $\zeta \neq 0$. Let $S^+ = \{s : \zeta_s > 0\}$, $S^- = \{s : \zeta_s < 0\}$. We have

$$\lambda^i \sum_s \pi^i_s u^{i\prime}(\omega^i_s + \sum_k R^i_k(s)\theta^i_k)\zeta_s > \lambda^i \left(a^i \sum_{s\in S^+} \pi^i_s\zeta_s + b^i \sum_{s\in S^-} \pi^i_s\zeta_s \geq 0 \right)$$

That means q is no-arbitrage for any agent i. Under A1, A2, P, U1bis, if there exists a no-arbitrage price then (see e.g. Werner [11], Page and Wooders [8], Dana, Le Van, Magnien [5]) there exist $[(\theta^{*i})_{i=0,\dots,L}; q^* >> 0]$ such that
(a) $\forall i$, θ^{*i} solves problem $(\mathcal{Q})$
(b) $\sum_{i=0}^L \theta^{*i} = 0$
Condition $(\mathcal{E})$ implies

$$(c)\ \sum_{i=0}^L \tau^{*i}_s c^{*i}_s = \sum_{i=0}^L \tau^{*i}_s \omega^i_s$$

where

$$c^{*i}_s = \omega^i_s + \sum_{k=1}^K R^i_k(s)\theta^{*i}_k.$$

We have an equilibrium $[(\theta^{*i})_{i=0,\dots,L}; (\tau^{*i}_s)_{i,s},\ q^* \neq 0]$.
Conversely, if $[(\theta^{*i})_{i=0,\dots,L}; (\tau^{*i}_s)_{i,s},\ q^* \neq 0]$ is an equilibrium then

$$\forall k',\ q^*_{k'} = \lambda^{*i} \sum_{s=1}^S \pi^i_s u^{i\prime} \left(\omega^i_s + \sum_{k=1}^K R^i_k(s)\theta^{*i}_k \right) R^i_{k'}(s),\ \lambda^{*i} > 0$$

and

$$\forall i, a^i < u^{i\prime}\left(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k^{*i}\right) < b^i$$

One can show as just above that $q^* \in \cap_i S^i$, i.e. a no-arbitrage price. ■

Comments

1. Condition $(\mathcal{E})$ means that for any portfolio $\theta_1, \ldots, \theta_k$, the return it yields will be the same for any country i if it is valued in currency 0. This condition is very important. We give an example where it is not satisfied and we have no equilibrium.

We consider an economy with two countries, 0 and 1, two states of nature and two assets. We assume

$$R^o = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad R^1 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

In this economy, condition $(\mathcal{E})$ is not satisfied. We have

$$\begin{aligned} W^0 &= \{(\theta_1, \theta_2) : \theta_1 \geq 0,\ \theta_1 + 2\theta_2 \geq 0\} \\ W^1 &= \{(\theta_1, \theta_2) : \theta_2 \geq 0,\ 2\theta_1 + \theta_2 \geq 0\} \\ S^0 &= \{(p_1, p_2) : p_1 > 0,\ p_2 > 0,\ 2p_1 - p_2 > 0\} \\ S^1 &= \{(p_1, p_2) : p_1 > 0,\ p_2 > 0,\ 2p_2 - p_1 > 0\} \end{aligned}$$

One can check that $(1, 1) \in S^0 \cap S^1$. From Proposition 3, there exist $[(\theta^{*i})_{i=0,1}; (q^*(1), q^*(2))]$ such that

(a) $\forall i,\ \theta^{*i}$ solves problem $(\mathcal{P})$

(b) $\sum_{i=0}^{2} \theta^{*i} = 0$

(c) $(q^*(1), q^*(2)) >> 0$

If there exists an equilibrium then

$$\tau_1^{*1} = -\frac{R_1^0(1)\theta_1^{*0} + R_2^0(1)\theta_2^{*0}}{R_1^1(1)\theta_1^{*1} + R_2^1(1)\theta_2^{*1}}$$

We obtain

$$\begin{aligned} \tau_1^{*1} &= \frac{\theta_1^{*1}}{\theta_2^{*1}} \\ &= -\frac{q_2^*}{q_1^*} < 0 \end{aligned}$$

since $q_1^*\theta_1^{*1} + q_2^*\theta_2^{*1} = 0$: a contradiction.

2. Consider condition $(\mathcal{E})$. We assume that for any country i, the asset i is riskless. The returns $R_i^i(s)$ will not depend on s and are assumed to be constant. Condition $(\mathcal{E})$ may be written as

$$Log\tau_s^{*i} = LogR_i^0(s) - LogR_i^i$$

Let $E_i^i = LogR_i^i$. Assume that the returns are given, as in Fontaine [6], relation (5)

$$LogR_i^0(s) = E_i^0(s) + \sum_{m=1}^{M} b_{im}^0 \tilde{f}_m^0(s)$$

where $\tilde{f}_m^0$ are the common factors, we then obtain

$$Log\tau_s^{*i} = E_i^0(s) - E_i^i + \sum_{m=1}^{M} b_{im}^0 \tilde{f}_m^0(s) \tag{3}$$

which is relation (9) in Fontaine [6].
More generally, assume that

$$LogR_k^j(s) = E_k^j(s) + \sum_{m=1}^{M} b_{km}^j \tilde{f}_m^j(s)$$

Let $r_{jk}^0(s) = Log(\tau_s^{*j} R_k^j(s))$. r_{jk}^0 is the return of asset k in country j valued in currency 0. We get:

$$r_{jk}^0(s) = E_k^j(s) + \sum_{m=1}^{M} b_{km}^j \tilde{f}_m^j(s) + E_i^0(s) - E_i^i + \sum_{m=1}^{M} b_{im}^0 \tilde{f}_m^0(s) \tag{4}$$

which corresponds to relation (11) in Fontaine [6]. If Relation 4 holds for any country j, for any asset k, we then have an equilibrium in the two-period consumption model. However, this condition is not sufficient, in general, for the wealth model.
3. An equilibrium price is given by

$$\begin{aligned} \forall i,\ \forall k,\ q_k^* &= \lambda^i \sum_{s=1}^{S} \pi_s^i u^{i\prime}(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k^{*i}) R_k^i(s) \\ &= \lambda^i \sum_{s=1}^{S} \pi_s^i u^{i\prime} \left(\omega_s^i + \sum_{k=1}^{K} \frac{R_k^0(s)\theta_k^{*i}}{\tau_s^{*i}} \right) \frac{R_k^0(s)}{\tau_s^{*i}} \end{aligned}$$

In particular, if the market is complete and non-redundant, i.e. the matrix $R_k^0(s)$ is square and invertible, we have:

$$\forall i,\ \lambda^i \pi_s^i u^{i\prime} \left(\omega_s^i + \sum_{k=1}^{K} \frac{R_k^0(s)\theta_k^{*i}}{\tau_s^{*i}} \right) \frac{1}{\tau_s^{*i}} = \lambda^0 \pi_s^0 u^{0\prime}(\omega_s^0 + \sum_{k=1}^{K} R_k^0(s)\theta_k^{*0})$$

The equilibrium price q^* depends on the expectations, the returns, the initial endowments and the equilibrium portfolio in country 0.

Observe that when the countries are risk-neutral ($u^i(x) = x$) then an equilibrium exists if and only if $\forall s,\ \forall i, \pi_s^i = \frac{\pi_s^0 \tau_s^{*i}}{\sum_\sigma \pi_\sigma^0 \tau_\sigma^{*i}}$.

4 On the Purchasing Power Parity (PPP)

Let q^* be an equilibrium price. We know that q^* is NA1. From Dana and Jeanblanc-Piqué [2], there exists $\big((\beta_s^i > 0); i = 0, \ldots, L; s = 1, \ldots, S\big)$ such that $\forall i,\ q^* = \sum_s \beta_s^i R^i(s)$. Define $p_s^{*i} = \beta_s^i,\ s = 1, \ldots, S; i = 0, \ldots, L$. We have

$$\forall i,\ q_k^* = \sum_s p_s^{*i} R_k^i(s) = \sum_s \frac{p_s^{*i}}{\tau_s^{*i}} R_k^0(s) = \sum_s p_s^{*0} R_k^0(s) \tag{5}$$

Let

$$Z = \{z \in \mathbb{R}^S : \sum_s z_s R_k^0(s) = 0,\ \forall k\}$$

$Z = \{0\}$ if the market is complete. From (5), we get

$$\forall i, p_s^{*i} = \tau_s^{*i}(p_s^{*0} + z_s^i)$$

with $(z^i) \in Z$. Define

$$\forall i \neq 0,\ \forall s,\ \tilde{p}_s^{*i} = p_s^{*i} - \tau_s^{*i} z_s^i = \tau_s^{*i} p_s^{*0} \tag{6}$$

$$\tilde{p}_s^{*0} = p_s^{*0} \tag{7}$$

We claim that $(\tilde{p}_s^{*i})$ is a prices system for the consumption good. Indeed, let

$$c_s^{*i} = \omega_s^i + \sum_k R_k^i(s)\theta_k^{*i},\ \forall i,\ \forall s$$

We have

$$\sum_s \tilde{p}_s^{*i} c_s^{*i} = \sum_s \tilde{p}_s^{*i} \omega_s^i + \sum_k q_k^* \theta_k^{*i} = \sum_s \tilde{p}_s^{*i} \omega_s^i$$

since $\sum_k q_k^* \theta_k^{*i} = 0$.

Observe that, for any portfolio of country i, θ^i,

$$q^* \cdot \theta^i = \sum_s \tilde{p}_s^{*i} (\sum_k R_k^i(s)\theta_k^i)$$

Now, let

$$\sum_s \pi_s^i u^i(\omega_s^i + \sum_k R_k^i(s)\theta_k^i) > \sum_s \pi_s^i u^i(\omega_s^i + \sum_k R_k^i(s)\theta_k^{*i})$$

This implies $q^* \cdot \theta^i > q^* \cdot \theta^{*i}$ or equivalently

$$\sum_s \tilde{p}_s^{*i} (\sum_k R_k^i(s)\theta_k^i) > \sum_s \tilde{p}_s^{*i} (\sum_k R_k^i(s)\theta_k^{*i})$$

And if we define

$$c_s^{*i} = \omega_s^i + \sum_k R_k^i(s)\theta_k^{*i},\ \forall i,\ \forall s$$

$$c_s^i = \omega_s^i + \sum_k R_k^i(s)\theta_k^i, \ \forall i, \ \forall s$$

we obtain

$$\sum_s \tilde{p}_s^{*i} c_s^{*i} > \sum_s \tilde{p}_s^{*i} c_s^i$$

That means $[(c_s^{*i}); (\tilde{p}_s^{*i}); i = 0, \ldots, L; s = 1, \ldots, S]$ is an equilibrium for the model where

(a) each agent i solves:

$$\max \sum_s \pi_s^i u^i(c_s^i)$$

under the constraints:

$$c^i \in X^i \quad = \quad \{c \in \mathbb{R}^S \mid \exists \theta \in \mathbb{R}^K, \ c_s = \omega_s^i + \sum_k R_k^i \theta_k\} \cap \mathbb{R}_+^S \text{ for the consumption model}$$
$$c^i \in X^i \quad = \quad \{c \in \mathbb{R}^S \mid \exists \theta \in \mathbb{R}^K, \ c_s = \omega_s^i + \sum_k R_k^i \theta_k\} \text{ for the wealth model}$$

and the budget constraint

$$\sum_s p_s^{*i} c_s^i \quad \leq \quad \sum_s p_s^{*i} \omega_s^i$$

and

(b) $\forall s, \ \sum_i \tau_s^{*i} c_s^i = \sum_i \tau_s^{*i} \omega_s^i$

Conversely, under A3, one can check that if $[(c_s^{*i}); (\tilde{p}_s^{*i}); i = 0, \ldots, L; s = 1, \ldots, S]$ is an equilibrium for the model given just above with $\tilde{p}_s^{*i} = \tau_s^{*i} \tilde{p}_s^{*0}, \ \forall i, \ \forall s$ then $[\theta^{*i}, q^*]$ is an equilibrium on the international asset market where

$$c_s^{*i} = \omega_s^i + \sum_k R_k^i(s)\theta_k^{*i}, \ \forall i, \ \forall s$$

and

$$q^* = \sum_s \tilde{p}_s^{*0} R^0(s).$$

Indeed, let

$$\sum_s \pi_s^i u^i(\omega_s^i + \sum_k R_k^i(s)\theta_k^i) > \sum_s \pi_s^i u^i(\omega_s^i + \sum_k R_k^i(s)\theta_k^{*i})$$

That implies

$$\sum_s p_s^{*i}(\omega_s^i + \sum_k R_k^i(s)\theta_k^i) > \sum_s p_s^{*i}(\omega_s^i + \sum_k R_k^i(s)\theta_k^{*i})$$

or equivalently

$$\sum_k (\sum_s p_s^{*i} R_k^i(s))\theta_k^i > \sum_k (\sum_s p_s^{*i} R_k^i(s))\theta_k^{*i}$$

i.e.

$$q^* \cdot \theta^i > q^* \cdot \theta^{*i}$$

It remains to show that the asset market clears. Since

$$\sum_i \tau_s^{*i} c_s^{*i} = \sum_i \tau_s^{*i} \omega_s^i$$

we have

$$\sum_i \sum_k \tau_s^{*i} R_k^i(s) \theta_k^{*i} = 0$$

or equivalently

$$\sum_k R_k^0(s)(\sum_i \theta_k^{*i}) = 0$$

Assumption A3 implies $\sum_i \theta_k^{*i} = 0, \ \forall k$.

We have proven

Proposition 12 *Let $[\theta^{*i}, q^*]$ be an equilibrium on the international asset market and let*

$$c_s^{*i} = \omega_s^i + \sum_k R_k^i(s) \theta_k^{*i}, \ \forall i, \ \forall s$$

Then there exists a price system $(\tilde{p}_s^{*i})_{i,s}$ such that $[(c_s^{*i}), (\tilde{p}_s^{*i})]$ is an equilibrium for the model where

(a) each agent i solves:

$$\max \sum_s \pi_s^i u^i(c_s^i)$$

under the constraints:

$$c^i \in X^i = \{c \in \mathbb{R}^S \mid \exists \theta \in \mathbb{R}^K, \ c_s = \omega_s^i + \sum_k R_k^i \theta_k\} \cap \mathbb{R}_+^S \text{ for the consumption model}$$
$$c^i \in X^i = \{c \in \mathbb{R}^S \mid \exists \theta \in \mathbb{R}^K, \ c_s = \omega_s^i + \sum_k R_k^i \theta_k\} \text{ for the wealth model}$$

and the budget constraint

$$\sum_s p_s^{*i} c_s^i \leq \sum_s p_s^{*i} \omega_s^i$$

and

(b) $\forall s, \ \sum_i \tau_s^{*i} c_s^i = \sum_i \tau_s^{*i} \omega_s^i$

Moreover the prices system $(\tilde{p}_s^{*i})_{i,s}$ satisfies the PPP, i.e., $\forall i, \ \forall s, \ \tilde{p}_s^{*i} = \tau_s^{*i} \tilde{p}_s^{*0}$

Conversely, under A3, if $[(c_s^{*i}); (\tilde{p}_s^{*i}); i = 0, \ldots, L; s = 1, \ldots, S]$ *is an equilibrium for the model given just above with* $\tilde{p}_s^{*i} = \tau_s^{*i} \tilde{p}_s^{*0}$, $\forall i$, $\forall s$ *then* $[\theta^{*i}, q^*]$ *is an equilibrium on the international asset market where*

$$c_s^{*i} = \omega_s^i + \sum_k R_k^i(s) \theta_k^{*i},\ \forall i,\ \forall s$$

and

$$q^* = \sum_s \tilde{p}_s^{*0} R^0(s).$$

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