Equilibrium in International Asset Market

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December 16, 2009

1 The Model

Let us consider a two-period economy with $L + 1$ countries and $K$ assets. We suppose there exists one consumption good which may be traded between the $L + 1$ countries. In each country there is only one consumer. In period 0, agent $i$, ($i = 0, \ldots, L$) purchases assets and consumes in period 1. There are $S$ states of nature in period 1. If state $s$ occurs, in period 1, the consumer in country $i$ will consume $c_i^s$:

$$c_i^s = \omega_i^s + \sum_{k=1}^{K} R_i^k(s) \theta_i^k$$

where $\theta_i^k$ is the portfolio she purchased in period 0, $\omega_i^s$ is the initial endowment of consumption good, $R_i^k(s) \geq 0$ is the return of asset $k$ in country $i$. The initial endowment $\omega_i^s$ and the return $R_i^k(s)$ are valued in currency of country $i$.

We make the following assumptions:

**A1:** For any $i$, any $s$, $\sum_{k=1}^{K} R_i^k(s) > 0$

**A2:** For any $i$, any $k$, $\sum_{s=1}^{S} R_i^k(s) > 0$

These assumptions are not very stringent. If **A1** is not satisfied for some $i$, some $s$, in this case, country $i$ will not make any exchange on the asset market in state $s$. If **A2** is not satisfied for some $i$, some $k$, country $i$ will never purchase asset $k$.

**P:** For every state $s$, every country $i$, $\pi_i^s > 0$, where $\pi_i^s$ is the belief probability of agent $i$ that state $s$ occurs in period 1.

For any state $s$, we take the consumption price of country 0 as numeraire. Hence, if $\tau_i^s$ is the exchange rate in state $s$ between countries $i$ and 0, the consumption of agent $i$ valued in currency 0 is

$$\tau_i^s c_i^s = \tau_i^s \omega_i^s + \sum_{k=1}^{K} \tau_i^s R_i^k(s) \theta_i^k$$
2 The consumption model

We first consider the two-period consumption model. Let \((\pi^i_s) = 0\) in the \(S\)-unit simplex be the belief of agent \(i\). If \(q\) is the asset price, agent \(i\) will solve:

\[
(P) \quad \max \sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s) \theta^i_k)
\]

\[\forall s, \quad \omega^i_s + \sum_{k=1}^{K} R^i_k(s) \theta^i_k \geq 0\]

\[\sum_{k=1}^{K} q^i_k \theta^i_k \leq 0.\]

We suppose that for any \(i\), agent \(i\) has no initial endowment for the assets. We also assume

\[U1:\] The utility function \(u^i\) is concave, strictly increasing, differentiable in \(\mathbb{R}^{++}\) for any \(i\).

**Definition 1**

An equilibrium is a list \([((\theta^{x^i}, (c^{x^i}; \tau^{x^i}))_{s=1,..,S})_{i=0,..,L}, q^x \neq 0]\) such that

1. \(\forall i, \theta^{x^i}\) will solve problem \((P)\) given \(q^x\)
2. \(\sum_{i=0}^{L} \theta^{x^i} = 0\)
3. \(\sum_{i=0}^{L} \tau^{x^i} \omega^i = \sum_{i=0}^{L} \tau^i \omega^i_s\).

Relation (2) is the market clearing on the asset market while relation (3) is the balance on the consumption good market valued in currency 0.

**Remark**

By definition, \(\tau^{x^i}_s = 1\) for any \(s\).

It is obvious that the consumption set \(X^i\) is

\[
X^i = \left\{ \theta \in \mathbb{R}^K : \text{for any } s, \quad \omega^i_s + \sum_{k=1}^{K} R^i_k(s) \theta^i_k \geq 0 \right\}
\]

**Definition 2**

\(w\) is a useful assets purchase for agent \(i\) if for any \(\lambda \geq 0\), for any \(\theta \in X^i\), one has:

\[(a) \quad \omega^i_s + \sum_{k=1}^{K} R^i_k(s)(\theta_k + \lambda w_k) \geq 0\]

\[(b) \quad \sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)(\theta_k + \lambda w_k) \geq \sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)\theta_k)\]

Let \(W^i\) denote the set of useful vectors for agent \(i\).
Proposition 1 We have
\[ W^i = \left\{ w \in \mathbb{R}^K : \sum_{k=1}^{K} R_k^i(s) w_k \geq 0, \ \forall s \right\} \]

Proof: Consider (a) in the previous definition. Divide the LHS by \( \lambda \) and let it go to infinity. We obtain
\[ \sum_{k=1}^{K} R_k^i(s) w_k \geq 0. \]
Conversely, assume \( \forall s, \sum_{k=1}^{K} R_k^i(s) w_k \geq 0. \) Then obviously, for any \( \theta \in X^i, \) any \( \lambda \geq 0, \) one has (a). From the increasingness of \( u^i, \) we have
\[ S \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i) + \sum_{k=1}^{K} R_k^i(s)(\theta_k + \lambda w_k) \geq \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i) + \sum_{k=1}^{K} R_k^i(s)\theta_k \]
We obtain (b). ■

Definition 3
A vector \( q \) is a no-arbitrage price for agent \( i \) if \( q \cdot w > 0, \) for all \( w \in W^i. \)
Let \( S^i \) denote the cone of no-arbitrage prices for agent \( i. \) Then, obviously,
\[ S^i = -\text{int}(W^i)^0. \]
In finance, there is another concept of no-arbitrage. We call it NA1. A vector \( q \) is a NA1 price, or more simply NA1, if for any country \( i, \) for any portfolio \( \theta \) which satisfies \( R_k^i(s) \cdot \theta \geq 0, \ \forall s, \) and \( R_k^i(s') \cdot \theta > 0 \) for some \( s', \) then we have \( q \cdot \theta > 0. \)
We introduce an assumption on the exchange rates:
\( (E) \) There exist \( [(\tau_{s_i}^i > 0); i = 1, \ldots, L; s = 1, \ldots, S], \tau_s^0 = 1; s = 1, \ldots, S \) such that
\[ \forall i \neq 0, \forall s, \forall k, \ \tau_{s_i}^i R_k^i(s) = R_k^0(s). \]

Proposition 2 Under \( (E), \) a vector \( q \) is NA1 iff, \( \forall s, \ R_k^0(s) \cdot \theta \geq 0 \) and \( R_k^0(s') \cdot \theta > 0 \) for some \( s', \) then \( q \cdot \theta > 0. \)
Proof: Obvious. ■

Proposition 3 Assume \( A1, \ A2, \ P, \ U1 \) and \( \omega_s^i > 0, \forall s, \ \forall i. \) Assume moreover the no-arbitrage condition
\[ (\text{NA}) \cap \bigcap_{i=0}^{L} S^i \neq \emptyset \]
Then there exist \( [(\theta^i)_{i=0,...,L}; q^* >> 0] \) such that
(a) \( \forall i, \theta^i \) solves problem \( (P) \)
(b) \( \sum_{i=0}^{L} \theta^i = 0 \)
Proof: The proof may be found in several papers, e.g., Werner [11], Page and Wooders [8], Dana, Le Van, Magnien [5]. The strict positivity of $q^*$ comes from the strict increasingness of the $u^i$ and assumptions $A1, A2$. 

We now introduce an assumption which ensures the non-emptiness of $S^i$.

**A3:** For any $i$, there exists no non-null $(\theta_1, \ldots, \theta_K)$ which satisfies

$$\forall s, \sum_{k=1}^{K} R^i_k(s) \theta_k = 0$$

This assumption means that, for any country $i$, the $K$ assets are not redundant. With this assumption, for any $i$, $W^i$ contains no line, and $S^i$ is non-empty.

**Proposition 4** Assume A3. Then $q$ is NA1 iff it is a no-arbitrage price.

**Proof:** Let $q$ be no-arbitrage. Given $i$, let $w$ satisfy $R^i_k(s) \cdot w \geq 0$, $\forall s$ and $R^i_k(s') \cdot w > 0$ for some $s'$. In this case $w \in W^i \setminus \{0\}$. Hence $q \cdot w > 0$. That means $q$ is NA1.

Conversely, let $q$ be NA1. Given $i$, let $w \in W^i \setminus \{0\}$. then we have $R^i_k(s) \cdot w \geq 0$, $\forall s$ and $R^i_k(s') \cdot w > 0$ for some $s'$. If not, $R^i_k(s) \cdot w = 0$, $\forall s$ and from A3, $w = 0$: a contradiction. Since $q$ is NA1, we have $q \cdot w > 0$, i.e. $q$ is no-arbitrage. 

**Proposition 5** (a) If $q^*$ is an equilibrium price then it is NA1.

(b) Assume A3. If $q^*$ is an equilibrium price then it is both NA1 and no-arbitrage.

**Proof:** (a) Given $i$, let $\psi$ satisfy $R^i_k(s) \cdot \psi \geq 0$, $\forall s$ and $R^i_k(s') \cdot \psi > 0$ for some $s'$. Let $\theta^{s^i}$ denote the associated equilibrium portfolio. Since $u^i$ is strictly increasing, and $\pi^i_s > 0$, $\forall s$, we have

$$\sum_{s=1}^{S} \pi^i_s u^i(\omega^i + \sum_{k=1}^{K} R^i_k(s)(\theta^{s^i}_k + \psi_k)) > \sum_{s=1}^{S} \pi^i_s u^i(\omega^i + \sum_{k=1}^{K} R^i_k(s)\theta^{s^i}_k)$$

That implies $q \cdot \psi > 0$.

(b) The result follows from (a) and Proposition 4. 

**Proposition 6** Assume $A1, A2, A3, P, U1$ and $E$. Assume that for any $i$, $\omega^i_s > 0, \forall s$. Then there exist $[(\theta^{s^i})_{i=0,\ldots,L}; q^* \neq 0]$ such that $[(\theta^{s^i})_{i=0,\ldots,L}; (\tau^{s^i})_{i,s}; q^* \neq 0]$ is an equilibrium.
Proof: Under (E), the set $W^i$

\[ W^i = \left\{ w \in \mathbb{R}^K : \sum_{k=1}^K R^i_k(s) w_k \geq 0, \forall s \right\} \]

is independent of $i$ and hence $S^i$ is the same for all $i$. We will show that $S^0$ is non-empty. Indeed, let $w \in W^0 \setminus \{0\}$. Then there exists $s'$ such that $\sum_{k=1}^K R^0_k(s') w_k > 0$. If not, we have: $\forall s, \sum_{k=1}^K R^0_k(s) w_k = 0$. From A3, $w = 0$ which is a contradiction. Now, let $q \in \mathbb{R}^K$ be defined by $\forall k, q_k = \sum_{s=1}^S R^0_k(s)$. Then $q \cdot w > 0$ for any $w \in W^0 \setminus \{0\}$. That means $q \in S^0$.

The No-Arbitrage condition (NA) is therefore satisfied.

From Proposition 3, there exist $[(\theta^* i)_{i=0,\ldots,L}; q^* \neq 0]$ such that

(a) $\forall i, \theta^* i$ solves problem (P)

(b) $\sum_{i=0}^L \theta^* i = 0$.

Condition (E) implies

(c) $\sum_{i=0}^L \tau^* i c^* = \sum_{i=0}^L \omega^* i$

where

$c^* = \omega^* i + \sum_{k=1}^K R^i_k(s) \theta^* i$.

We end the proof. 

3 The wealth model

We drop the constraints

$\forall s, \omega^* i + \sum_{k=1}^K R^i_k(s) \theta^* i \geq 0$

We replace U1 by

U1bis: For every $i$, the utility function is is concave, strictly increasing, differentiable in $\mathbb{R}$.
Let $a^i = u^i(+\infty)$, $b^i = u^i(-\infty)$, $i = 0, \ldots, L$. Let $q$ be the asset price. Country $i$ will solve:

$$
(Q) \quad \max \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s) \theta_k^i) + \sum_{k=1}^{K} q_k \theta_k^i \leq 0.
$$

We suppose, as in the two-period consumption model, that for any $i$, agent $i$ has no initial endowment for the assets.

**Definition 4**

An equilibrium is a list $[(\theta_s^i, (c_s^i, \tau_s^i))_{s=1,\ldots,S})_{i=0,\ldots,L}, q^* \neq 0]$ such that

1. $\forall i$, $\theta_s^i$ will solve problem $(Q)$ given $q^*$
2. $\sum_{i=0}^{L} \theta_s^i = 0$
3. $\sum_{i=0}^{L} \tau_s^i = \sum_{i=0}^{L} \tau_s^i \omega_s^i$.

Relation (2) is the market clearing on the asset market while relation (3) is the balance on the consumption good market valued in currency 0.

**Remark** By definition, as before, $\tau_s^0 = 1$ for any $s$.

It is obvious that the consumption set $X^i$ is now $\mathbb{R}^K$.

In order to prove existence of equilibrium, we will introduce and characterize the useful vectors and the no-arbitrage prices.

**Definition 5**

$w$ is a useful assets purchase for agent $i$ if for any $\lambda \geq 0$, for any $\theta \in X^i$, one has:

$$
\sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s) \theta_k^i + \lambda w^i_k) \geq \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s) \theta_k^i) \quad \forall \lambda \geq 0.
$$

From Rockafellar [10], $w$ is useful for agent $i$, if, and only if, there exists $\theta \in X^i$ such that

$$
\sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s) \theta_k^i + \lambda w^i_k) \geq \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s) \theta_k^i), \forall \lambda \geq 0.
$$

Let $W^i$ denote the set of useful vectors for agent $i$. We will characterize it.

**Proposition 7** A vector $w$ is useful for $i$ iff:

$$
\forall \theta \in \mathbb{R}^K, \sum_{k'=1}^{K} w_{k'} \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s) \theta_k^i) R_{k'}^i(s) \geq 0 \quad (1)
$$

**Proof:** It is very similar to those given in Dana and Le Van [3], [4] by using the concavity and the differentiability of the $u^i$. ■
We can have another characterization of $W^i$. The proof of the following
proposition is adapted from Dana and Le Van [4].

**Proposition 8** Let $w \in X^i$ and let $\zeta_s = \sum R_i^k(s)w_k, \ \forall s$, $S^+ = \{ s : \zeta_s > 0 \}$,
$S^- = \{ s : \zeta_s < 0 \}$. The vector $w$ is useful for $i$ iff

$$a^i \sum_{s \in S^+} \pi^i_s \zeta_s + b^i \sum_{s \in S^-} \pi^i_s \zeta_s \geq 0 \quad (2)$$

**Proof:** From Proposition 7, $w$ is useful iff for any $\theta \in \mathbb{R}^K$, we have

$$\sum_{s=1}^S \pi^i_s u^i \left( \omega^i_s + \sum_{k=1}^K R^i_k(s)(\theta_k + \lambda w_k) \right) \geq \sum_{s=1}^S \pi^i_s u^i (\omega^i_s + \sum_{k=1}^K R^i_k(s)\theta_k), \ \forall \lambda \geq 0.$$  

Take $\theta = 0$. We then have

$$\sum_{s=1}^S \pi^i_s u^i (\omega^i_s + \lambda \zeta_s) \geq \sum_{s=1}^S \pi^i_s u^i (\omega^i_s), \ \forall \lambda \geq 0.$$ 

Thus, $\zeta$ is useful for the function $(c_s)_s \rightarrow \sum \pi^i_s u^i(c_s)$. We then have for any $(c_s)_s$

$$0 \geq \sum_{s=1}^S \pi^i_s u^i(c_s) - \sum_{s=1}^S \pi^i_s u^i(c_s + \zeta_s) \geq -\sum_{s=1}^S \pi^i_s u^i(c_s)\zeta_s.$$ 

This implies $\sum_{s=1}^S \pi^i_s u^i(c_s)\zeta_s \geq 0$. For any $s \in S^+$ let $c_s$ go to $+\infty$, and for $s \in S^-$, let $c_s$ go $-\infty$. We then obtain (2).

The converse is obvious since $u''$ is non-increasing. ■

**Corollary 1** If $a^i = 0$ or $b^i = +\infty$ then $W^i = \{ w \in \mathbb{R}^K : \sum_{k=1}^K R^i_k(s)w_k \geq 0, \ \forall s \}$

**Proof:** It is obvious. ■

As before, we define no-arbitrage prices.

**Definition 6**
A vector $q$ is a no-arbitrage price for agent $i$ if $q \cdot w > 0$, for all $w \in W^i$. A vector $q$ is called no-arbitrage price if it is for every $i$.
Let $S^i$ denote the cone of no-arbitrage prices for agent $i$. Then, obviously, $S^i = -\text{int}(W^i)^0$.
In finance, there is another concept of no-arbitrage. We call it NA1. A vector $q$ is a NA1 price, or more simply NA1, if for any country $i$, for any portfolio $\theta$ which satisfies $R^i_k(s) \cdot \theta \geq 0, \ \forall s$, and $R^i_k(s') \cdot \theta > 0$ for some $s'$, then we have $q \cdot \theta > 0$.  

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Proposition 9  (a) If \( q \) is no-arbitrage, then it is NA1. If \( q^* \) is an equilibrium price, then it is NA1.
(b) Assume A3. If \( u^i \) is strictly concave then

\[
\sum_{s} \pi^i_s u^i(\omega^i_s + \sum_{k} R^i_k(s)(\theta^i_k + w_k)) > \sum_{s} \pi^i_s u^i(\omega^i_s + \sum_{k} R^i_k(s)\theta^i_k)
\]

for any \( \theta \), any \( w \in \mathcal{W}^i \setminus \{0\} \). And any equilibrium price is no-arbitrage.

Proof: (a)Let \( q \) be no-arbitrage. Given \( i \), let \( w \) satisfy \( R^i_k(s) \cdot w \geq 0 \), \( \forall s \) and \( R^i_k(s') \cdot w > 0 \) for some \( s' \). In this case \( w \in \mathcal{W}^i \setminus \{0\} \). Hence \( q \cdot w > 0 \). That means \( q \) is NA1.
Given \( i \), let \( \psi \) satisfy \( R^i_k(s) \cdot \psi \geq 0 \), \( \forall s \) and \( R^i_k(s') \cdot \psi > 0 \) for some \( s' \). Let \( \theta^w \) denote the associated equilibrium portfolio. Since \( u^i \) is strictly increasing, and \( \pi^i_s > 0 \), \( \forall s \), we have

\[
\sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)(\theta^i_k + \psi_k)) > \sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)\theta^i_k)
\]

That implies \( q \cdot \psi > 0 \).

(b) Let \( w \in \mathcal{W}^i \setminus \{0\} \). Then from A3, \( \sum_{s} R^i_k(s)w_k \neq 0 \). If

\[
\sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)(\theta^i_k + w_k)) = \sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)\theta^i_k)
\]

then

\[
\sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)(\theta^i_k + \frac{1}{2}w_k)) > \sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)(\theta^i_k + w_k))
\]

which is a contradiction since

\[
\sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)(\theta^i_k + w_k)) \geq \sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)(\theta^i_k + \frac{1}{2}w_k))
\]

Let \([\theta^w, q^*] \) be an equilibrium. Then for any \( w \in \mathcal{W}^i \setminus \{0\} \) we have

\[
\sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)(\theta^i_k + w_k)) > \sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)\theta^i_k)
\]

This implies \( q^* \cdot w > 0 \).  ■

We have the first result for existence of equilibrium.

Proposition 10  Assume A1, A2, A3, P, U1bis, condition \( (\mathcal{E}) \), and for any \( i \), either \( a^i = 0 \) or \( b^i = +\infty \). Then there exists an equilibrium.
Proof: In this case, for any $i$, $W^i = \{ w \in \mathbb{R}^K : \sum_k R^i_k(s)w_k \geq 0, \forall s \}$. The proof is the same as for Proposition 6. 

More generally,

**Proposition 11** Assume $A1, A2, A3, P, U1bis, \text{condition (E)}$, and for any $i$, $a^i < u^i(\omega^i_s + \sum_{k=1}^K R^i_k(s)\theta^i_k) < b^i, \forall \theta$.

Then there exists an equilibrium iff there exists a no-arbitrage price, i.e. there exist $[(\theta^i, \lambda^i > 0)_{i=0,...,L}]$ such that

$$\forall i, \forall j, \forall k', \lambda^i \sum_s \pi^i_s u^i(\omega^i_s + \sum_k R^i_k(s)\theta^i_k) R^i_{k'} = \lambda^j \sum_s \pi^j_s u^j(\omega^j_s + \sum_k R^j_k(s)\theta^j_k) R^j_{k'}$$

Proof: (1) Assume there exist $[(\theta^i, \lambda^i > 0)_{i=0,...,L}]$ such that

$$\forall i, \forall j, \forall k', \lambda^i \sum_s \pi^i_s u^i(\omega^i_s + \sum_k R^i_k(s)\theta^i_k) R^i_{k'} = \lambda^j \sum_s \pi^j_s u^j(\omega^j_s + \sum_k R^j_k(s)\theta^j_k) R^j_{k'}$$

Let

$$q_{k'} = \lambda^i \sum_s \pi^i_s u^i(\omega^i_s + \sum_k R^i_k(s)\theta^i_k) R^i_{k'}, \forall k'$$

We will show that $q$ is no-arbitrage. Indeed, let $w \in W^i \setminus \{0\}$. Let $\zeta_s = \sum_k R^i_k(s)w_k, \forall s$. We will show

$$q \cdot w = \lambda^i \sum_s \pi^i_s u^i(\omega^i_s + \sum_k R^i_k(s)\theta^i_k)\zeta_s > 0$$

From $A3$, $\zeta \neq 0$. Let $S^+ = \{ s : \zeta_s > 0 \}$, $S^- = \{ s : \zeta_s < 0 \}$. We have

$$\lambda^i \sum_s \pi^i_s u^i(\omega^i_s + \sum_k R^i_k(s)\theta^i_k)\zeta_s > \lambda^i \left( a^i \sum_{s \in S^+} \pi^i_s \zeta_s + b^i \sum_{s \in S^-} \pi^i_s \zeta_s \geq 0 \right)$$

That means $q$ is no-arbitrage for any agent $i$. Under $A1, A2, P, U1bis$, if there exists a no-arbitrage price then (see e.g. Werner [11], Page and Wooders [8], Dana, Le Van, Magnien [5]) there exist $[(\theta^{si})_{i=0,...,L}; q^* > 0]$ such that

(a) $\forall i$, $\theta^{si}$ solves problem (Q)

(b) $\sum_{i=0}^L \theta^{si} = 0$

Condition (E) implies

$$\sum_{i=0}^L \tau^{si}_s c^{si}_s = \sum_{i=0}^L \tau^{si}_s \omega^i_s$$

where

$$c^{si}_s = \omega^i_s + \sum_{k=1}^K R^i_k(s)\theta^{si}_k.$$
and
\[ \forall i, a^i < u'' \left( \omega^i + \sum_{k=1}^{K} R^i_k(s)\theta^i_k \right) < b^i \]

One can show as just above that \( q^* \in \cap_i S^i \), i.e. a no-arbitrage price. ■

**Comments**

1. Condition \((E)\) means that for any portfolio \( \theta_1, \ldots, \theta_k \), the return it yields will be the same for any country \( i \) if it is valued in currency 0. This condition is very important. We give an example where it is not satisfied and we have no equilibrium.

We consider an economy with two countries, 0 and 1, two states of nature and two assets. We assume
\[
R^0 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad R^1 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}
\]

In this economy, condition \((E)\) is not satisfied. We have
\[
W^0 = \{ (\theta_1, \theta_2) : \theta_1 \geq 0, \theta_1 + 2\theta_2 \geq 0 \}
\]
\[
W^1 = \{ (\theta_1, \theta_2) : \theta_2 \geq 0, 2\theta_1 + \theta_2 \geq 0 \}
\]
\[
S^0 = \{ (p_1, p_2) : p_1 > 0, p_2 > 0, 2p_1 - p_2 > 0 \}
\]
\[
S^1 = \{ (p_1, p_2) : p_1 > 0, p_2 > 0, 2p_2 - p_1 > 0 \}
\]

One can check that \((1, 1) \in S^0 \cap S^1\). From Proposition 3, there exist \(\{(\theta^* i)_i=0,1; (q^*(1), q^*(2))\}\) such that
(a) \( \forall i, \theta^* i \) solves problem \((P)\)
(b) \( \sum_{i=0}^{2} \theta^* i = 0 \)
(c) \( (q^*(1), q^*(2)) >> 0 \)

If there exists an equilibrium then
\[
\tau^*_1 = -\frac{R^0(1)\theta^*_1 + R^0(1)\theta^*_0}{R^1(1)\theta^*_0 + R^1(2)\theta^*_1}
\]

We obtain
\[
\tau^*_1 = \frac{\theta^*_1}{\theta^*_2} = \frac{-q^*_2}{q^*_1} < 0
\]

since \( q^*_1 \theta^*_1 + q^*_2 \theta^*_2 = 0 \): a contradiction.

2. Consider condition \((E)\). We assume that for any country \( i \), the asset \( i \) is riskless. The returns \( R^i(s) \) will not depend on \( s \) and are assumed to be constant. Condition \((E)\) may be written as
\[
\log \tau^*_s = \log R^0_i(s) - \log R^i_i
\]
Let \( E^i = \log R^i \). Assume that the returns are given, as in Fontaine [6], relation (5)

\[
\log R^0_i(s) = E^0_i(s) + \sum_{m=1}^{M} b^0_{im} \tilde{f}_m(s)
\]

where \( \tilde{f}_m \) are the common factors, we then obtain

\[
\log \tau^{*i} = E^0_i(s) - E^i_i + \sum_{m=1}^{M} b^0_{im} \tilde{f}_m(s)
\]

which is relation (9) in Fontaine [6].

More generally, assume that

\[
\log R^j_k(s) = E^j_k(s) + \sum_{m=1}^{M} b^j_{km} \tilde{f}_m(s)
\]

Let \( r^0_{jk}(s) = \log(\tau^{*j} R^j_k(s)) \). \( r^0_{jk} \) is the return of asset \( k \) in country \( j \) valued in currency 0. We get:

\[
r^0_{jk}(s) = E^j_k(s) + \sum_{m=1}^{M} b^j_{km} \tilde{f}_m(s) + E^0_i(s) - E^i_i + \sum_{m=1}^{M} b^0_{im} \tilde{f}_m(s)
\]

which corresponds to relation (11) in Fontaine [6]. If Relation 4 holds for any country \( j \), for any asset \( k \), we then have an equilibrium in the two-period consumption model. However, this condition is not sufficient, in general, for the wealth model.

3. An equilibrium price is given by

\[
\forall i, \forall k, q^*_k = \lambda^i \sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s) \theta^i_k) R^i_k(s)
\]

\[
= \lambda^i \sum_{s=1}^{S} \pi^i_s u^i \left( \omega^i_s + \sum_{k=1}^{K} \frac{R^0_k(s) \theta^i_k}{\tau^i_s} \right) \frac{R^0_k(s)}{\tau^i_s}
\]

In particular, if the market is complete and non-redundant, i.e. the matrix \( R^0_k(s) \) is square and invertible, we have:

\[
\forall i, \lambda^i \pi^i_s u^i \left( \omega^i_s + \sum_{k=1}^{K} \frac{R^0_k(s) \theta^i_k}{\tau^i_s} \right) \frac{1}{\tau^i_s} = \lambda^0 \pi^0_s u^0(\omega^0_s + \sum_{k=1}^{K} R^0_k(s) \theta^0_k)
\]

The equilibrium price \( q^* \) depends on the expectations, the returns, the initial endowments and the equilibrium portfolio in country 0.

Observe that when the countries are risk-neutral \( (u^i(x) = x) \) then an equilibrium exists if and only if \( \forall s, \forall i, \pi^i_s = \frac{\omega^i_s}{\sum_{\sigma} \pi^\sigma_s} \).
4 On the Purchasing Power Parity (PPP)

Let \( q^* \) be an equilibrium price. We know that \( q^* \) is NA1. From Dana and Jeanblanc-Piqué [2], there exists \( (\beta_s^i > 0; i = 0, \ldots, L; s = 1, \ldots, S) \) such that \( \forall i, q^* = \sum_s \beta_s^i R^i(s) \). Define \( p^*_s = \beta_s^i, s = 1, \ldots, S; i = 0, \ldots, L \). We have

\[
\forall i, \quad q^*_k = \sum_s p^*_s R^i_k(s) = \sum_s \frac{p^*_s}{\tau^*_s} R^0_k(s) = \sum_s p^*_s R^0_k(s) \tag{5}
\]

Let

\[
Z = \{ z \in \mathbb{R}^S : \sum_s z_s R^0_k(s) = 0, \forall k \}
\]

\( Z = \{ 0 \} \) if the market is complete. From (5), we get

\[
\forall i, \quad p^*_s = \tau^*_s (p^*_s + z^i)
\]

with \( (z^i) \in Z \). Define

\[
\forall i \neq 0, \forall s, \quad \tilde{p}^*_s = p^*_s - \tau^*_s z^i = \tau^*_s p^*_s + \tau^*_s z^i \tag{6}
\]

\[
\tilde{p}^*_0 = p^*_0 \tag{7}
\]

We claim that \( (\tilde{p}^*_s) \) is a prices system for the consumption good. Indeed, let

\[
c^*_s = \omega^i + \sum_k R^i_k(s) \theta^i_k, \forall i, \forall s
\]

We have

\[
\sum_s \tilde{p}^*_s c^*_s = \sum_s \tilde{p}^*_s \omega^i + \sum_k q^*_k \theta^i_k = \sum_s \tilde{p}^*_s \omega^i
\]

since \( \sum_k q^*_k \theta^i_k = 0 \).

Observe that, for any portfolio of country \( i \), \( \theta^i \),

\[
q^* \cdot \theta^i = \sum_s \tilde{p}^*_s (\sum_k R^i_k(s) \theta^i_k)
\]

Now, let

\[
\sum_s \pi^i u^i(\omega^i + \sum_k R^i_k(s) \theta^i_k) > \sum_s \pi^i u^i(\omega^i + \sum_k R^i_k(s) \theta^i_k)
\]

This implies \( q^* \cdot \theta^i > q^* \cdot \theta^i \) or equivalently

\[
\sum_s \tilde{p}^*_s (\sum_k R^i_k(s) \theta^i_k) > \sum_s \tilde{p}^*_s (\sum_k R^0_k(s) \theta^i_k)
\]

And if we define

\[
c^*_s = \omega^i + \sum_k R^i_k(s) \theta^i_k, \forall i, \forall s
\]
\[ c^i_s = \omega^i_s + \sum_k R^i_k(s)\theta^i_k, \forall i, \forall s \]

we obtain
\[ \sum_s p^s_i c^i_s > \sum_s p^s_i c^i_s \]

That means \([c^i_s; (\tilde{p}^s_i); i = 0, \ldots, L; s = 1, \ldots, S]\) is an equilibrium for the model where

(a) each agent \(i\) solves:

\[ \max \sum_s \pi^i_s u^i(c^i_s) \]

under the constraints:

\[ c^i \in X^i = \{ c \in \mathbb{R}^S \mid \exists \theta \in \mathbb{R}^K, c_s = \omega^i_s + \sum_k R^i_k(\theta_k) \cap \mathbb{R}_+^S \text{ for the consumption model} \]

\[ c^i \in X^i = \{ c \in \mathbb{R}^S \mid \exists \theta \in \mathbb{R}^K, c_s = \omega^i_s + \sum_k R^i_k(\theta_k) \} \text{ for the wealth model} \]

and the budget constraint
\[ \sum_s p^s_i c^i_s \leq \sum_s p^s_i \omega^i_s \]

and

(b) \(\forall s, \sum_i \tau^i_s c^i_s = \sum_i \tau^i_s \omega^i_s\)

Conversely, under A3, one can check that if \([c^*_s; (\tilde{p}^*_s); i = 0, \ldots, L; s = 1, \ldots, S]\) is an equilibrium for the model given just above with \(\tilde{p}^*_s = \tau^*_s \tilde{p}^0_s\), \forall i, \forall s \text{ then } [\theta^*_s, q^*] \text{ is an equilibrium on the international asset market where} \]

\[ c^*_s = \omega^i_s + \sum_k R^i_k(s)\theta^*_k, \forall i, \forall s \]

and

\[ q^* = \sum_s \tilde{p}^*_s R^0(s). \]

Indeed, let
\[ \sum_s \pi^i_s u^i(\omega^i_s + \sum_k R^i_k(s)\theta^i_k) > \sum_s \pi^i_s u^i(\omega^i_s + \sum_k R^i_k(s)\theta^i_k) \]

That implies
\[ \sum_s p^s_i (\omega^i_s + \sum_k R^i_k(s)\theta^i_k) > \sum_s p^s_i (\omega^i_s + \sum_k R^i_k(s)\theta^i_k) \]

or equivalently
\[ \sum_k \sum_s p^s_i R^i_k(s)\theta^i_k > \sum_k \sum_s p^s_i R^i_k(s)\theta^i_k \]
i.e. 
\[ q^* \cdot \theta^i > q^* \cdot \theta^{*i} \]

It remains to show that the asset market clears. Since 
\[ \sum_i \tau^s_i c^s_i = \sum_i \tau^s_i \omega^i_s \]
we have 
\[ \sum_i \sum_k \tau^s_i R^i_k(s) \theta^{*i}_k = 0 \]
or equivalently 
\[ \sum_k R^0_k(s) (\sum_i \theta^{*i}_k) = 0 \]

Assumption A3 implies \( \sum_i \theta^{*i}_k = 0, \forall k. \)

We have proven

Proposition 12 Let \( [\theta^{*i}, q^*] \) be an equilibrium on the international asset market and let 
\[ c^s_i = \omega^i_s + \sum_k R^i_k(s) \theta^{*i}_k, \forall i, \forall s \]

Then there exists a price system \((\tilde{p}^{*i}_s)_{i,s}\) such that \([(c^s_i), (\tilde{p}^{*i}_s)]\) is an equilibrium for the model where

(a) each agent \( i \) solves:

\[ \max \sum_s \pi^i_s u^i(c^s_i) \]

under the constraints:

\[ c^i \in X^i = \{ c \in \mathbb{R}^S \mid \exists \theta \in \mathbb{R}^K, c_s = \omega^i_s + \sum_k R^i_k(s) \theta_k \} \cap \mathbb{R}^S_+ \] for the consumption model
\[ c^i \in X^i = \{ c \in \mathbb{R}^S \mid \exists \theta \in \mathbb{R}^K, c_s = \omega^i_s + \sum_k R^i_k(s) \theta_k \} \] for the wealth model

and the budget constraint
\[ \sum_s p^{*i}_s c^i_s \leq \sum_s p^{*i}_s \omega^i_s \]

and
\[ \forall s, \sum_i \tau^{s_i} c^i_s = \sum_i \tau^{s_i} \omega^i_s \]

Moreover the prices system \((\tilde{p}^{*i}_s)_{i,s}\) satisfies the PPP, i.e., \( \forall i, \forall s, \tilde{p}^{*i}_s = \frac{\tau^{s_i} \tilde{p}^{0s}}{\tau^{s_i}} \)
Conversely, under A3, if \( [(c_s^i); (\tilde{p}_s^i); i = 0, \ldots, L; s = 1, \ldots, S] \) is an equilibrium for the model given just above with \( \tilde{p}_s^i = \tau_s^i \tilde{p}_s^0 \), \( \forall i, \forall s \) then \([\theta^i, q^*]\) is an equilibrium on the international asset market where
\[
c_s^i = \omega_s^i + \sum_k R_k(s)\theta^i_k, \forall i, \forall s
\]
and
\[
q^* = \sum_s \tilde{p}_s^0 R(s).
\]

References


